SURA'S

Mathematics

11th Standard

VOLUME - I & II

Based on the Updated New Textbook for 2019

Salient Features

- Prepared as per the updated new textbook for the year 2019.
- Exhaustive Additional Questions & Answers in all chapters.
- Govt. Model Question Paper-2018 [Govt. MQP-2018], First Mid-Term Test (2018) [First Mid-2018], Quarterly Exam - 2018 [QY-2018], Half Yearly Exam - 2018 [HY-2018], March Question Paper - 2019[March - 2019] are incorporated at appropriate sections.
- Govt. Model Question Paper - 1 and 2 with Answer Key.
- Sura's Model Question Paper - 1, 2 with Answer Key.
- March 2019 Question Paper with Answer Key.

SURA PUBLICATIONS

Chennai

Volume - I

MATHEMATICS

11th Standard

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01 Sets Relations and Functions

MUST KNOW DEFINITIONS

A set is a collection of well defined objects.

Type of sets

Empty set	:	A set containing no element.
Finite set	:	The number of elements in the set is finite.
Infinite set	:	The number of elements in the set is not finite.
Singleton set	:	A set containing only one element.
Equivalent set	:	Two sets having same number of elements.
Equal sets	:	Two sets exactly having the same elements.
Subset	:	A set X is a subset of Y if every element of X is also an element of Y. ($X \subseteq Y$)
Proper subset	:	X is a proper subset of Y if $X \subseteq Y$ and $X \neq Y$.
Power set	:	The set of all subsets of A is the power set of A.
Universal set	:	The set contains all the elements under consideration

Algebra of sets

Union	:	The union of two sets A and B is the set of elements which are either in A or in B ($A \cup B$)
Intersection	:	The intersection of two sets A and B is the set of all elements common to both A and B ($A \cap B$).
Complement of a set	:	The set of all elements of U (Universal set) that are not elements of A. (A') Set different($A \backslash B$) or ($A - B$)
		The difference of the two sets A and B is the set of all elements belonging to A but not to B
Disjoint sets	:	Two sets A and B are said to be disjoint if there is no element common to both A and B.
Open interval	:	The set $\{x: a < x < b\}$ is called an open interval and denoted by (a, b)
Closed interval	:	The set $\{x: a \leq x \leq b\}$ is called a closed interval and denoted by $[a, b]$

Neighbourhood of a point : Let a be any real number. Let $\in > 0$ be arbitrarily small real number. Then $(a - \in, a + \in)$ is called an "$\in$" neighbourhood of the point a and denoted by $N_{a,\in}$

Cartesian product of sets : The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of A and B and is denoted by $A \times B$.

Types of relation

Reflexive : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Symmetric : A relation R on a set A is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

Transitive : A relation R on a set A is said to be transitive if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Antisymmetric : A relation R on a set A is said to be anti-symmetric if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Equivalent : A relation R on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.

Function : A function f from a set A to a set B is a rule which assigns to each element of A, a unique element of B.
If $f: A \to B$, then A is the domain, B is the co-domain.

Types of algebraic functions

Identity function : A function that associates each real number to itself.

Absolute value function : The function $f(x)$ defined by $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Constant function : A function $f(x)$ defined by $f(x) = k$ where k is a real number.

Greatest integer function : The function $f: R \to R$ defined by $f(x) = [x]$ for all $x \in R$.

Signum function : The function f defined by $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Polynomial function : The function $f: R \to R$ defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$ where $a_0, a_1, \ldots, a_n$ are constants.

Rational function : The function $f: R \to R$ defined by $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and $p(x), q(x)$ are polynomial.

Algebra of functions

Addition : If $f: D_1 \to R$ and $g: D_2 \to R$, then their sum $f + g: D_1 \cap D_2 \to R$ such that $(f + g)(x) = f(x) + g(x)$ for all $x \in D_1 \cap D_2$.

Subtraction	:	If f_1: $D_1 \to R$ and g: $D_2 \to R$, then their difference $f - g$: $D_1 \cap D_2 \to R$ such that $(f - g)(x) - f(x) + g(x)$ for all $x \in D_1 \cap D_2$.
Product	:	If f_1: $D_1 \to R$ and g: $D_2 \to R$, then their product $f \cdot g$: $D_1 \cap D_2 \to R$ such that $(fg)(x) = f(x) + g(x)$ for all $x \in D_1 \cap D_2$.
Quotient	:	If f_1: $D_1 \to R$ and g: $D_2 \to R$, then their quotient $\frac{f}{g} : D_1 \cap D_2 - \{x : g(x) = 0\} \to R$ such that $\left(\frac{f}{2}\right)(x) = \frac{f(x)}{g(x)}$ such that for all $x \in D_1 \cap D_2 - \{x : g(x) = 0\}$.
Composition of functions	:	If f: $A \to B$ and g: $B \to C$ then $gof : A \to C$ defined by $gof(x) = g[f(x)]$ for all $x \in$ A.

Kinds of functions

One-one	:	A function f: $A \to B$ is said to be a one-one function (injection) if different elements of a have different images in B.
Onto	:	A function f: $A \to B$ is said to be an onto (surjection) function if every element of B is the image of some element of A.
Bijection	:	A function f: $A \to B$ is a bijection if one-one as well as onto.
Inverse of a function	:	Let f: $A \to B$ be a bijection. Then g: $B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f.

Formuale to remember

Demorgan's laws	:	1. $(A \cup B)' = A' \cap B'$ 2. $(A \cap B) = A' \cup B'$ 3. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$ 4. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$
Reflexive	:	aRa for all $a \in A$
Symmetric	:	aR$b \Rightarrow b$Ra for all $a, b \in A$
Transitive	:	aRb, bR$c \Rightarrow a$Rc for all $a, b, c \in A$
Antisymmetric	:	aRb and bR$a \Rightarrow a = b$ for all $a, b \in A$ $A \Delta B = (A \backslash B) \cup (B \backslash A)$
One-one function	:	If f: $A \to A$ then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in A$
Onto function	:	Co-domain = Range. If a set has n elements, then total number of subsets is 2^n.

TEXTUAL QUESTIONS

Exercise 1.1

1. Write the following in roster form.

(i) $\{x \in \mathbb{N} : x^2 < 121$ and x is a prime$\}$.

(ii) the set of all positive roots of the equation $(x-1)(x+1)(x^2-1)=0$.

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

(iv) $\left\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

Solution :

(i) $\{x \in \mathbb{N} : x^2 < 121$ and x is a prime$\}$.

Let A = $\{x \in \mathbb{N}: x^2 < 121$, and x is a prime$\}$

A = $\{2, 3, 5, 7\}$.

(ii) the set of all positive roots of the equation $(x-1)(x+1)(x^2-1)=0$.

Let B = {the set of positive roots of the equation $(x-1)(x+1)(x^2-1)=0$}

$\Rightarrow \quad x = 1, -1$

B = $\{1\}$.

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

Let C = $\{x \in \mathbb{N}: 4x + 9 < 52\}$

$\Rightarrow$ C = $\{x \in \mathbb{N}: 4x < 52 - 9\}$

$\Rightarrow$ C = $\{x \in \mathbb{N}: 4x < 43\}$

$\Rightarrow$ C = $\left\{x \in \mathbb{N}: x < \frac{43}{4}\right\}$

$\Rightarrow$ C = $\{x \in \mathbb{N}: x < 10.75\}$

$\Rightarrow$ C = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(iv) $\left\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$.

Let D = $\left\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

$\Rightarrow$ D = $\{x : x - 4 = 3x + 6, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{x : -4 - 6 = 3x - x, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{x : 2x = -10, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{x : x = -5, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{-5\}$

2. Write the set $\{-1, 1\}$ in set builder form.

Solution : Let P = $\{-1, 1\}$

$\Rightarrow$ P = $\{x : x$ is a root of $x^2 - 1 = 0\}$

3. State whether the following sets are finite or infinite.

(i) $\{x \in \mathbb{N} : x$ is an even prime number$\}$

(ii) $\{x \in \mathbb{N} : x$ is an odd prime number$\}$

(iii) $\{x \in \mathbb{Z} : x$ is even and less than 10$\}$

(iv) $\{x \in \mathbb{R} : x$ is a rational number$\}$

(v) $\{x \in \mathbb{N} : x$ is a rational number$\}$

Solution :

(i) $\{x \in \mathbb{N} : x$ is an even prime number$\}$

Let A = $\{x \in \mathbb{N}: x$ is an even prime number$\}$

$\Rightarrow$ A = $\{2\}$ $\Rightarrow$ A is a finite set.

(ii) $\{x \in \mathbb{N} : x$ is an odd prime number$\}$

Let B = $\{x \in \mathbb{N}: x$ is an odd prime number$\}$

$\Rightarrow$ B = $\{3, 5, 7, 11, \ldots\}$

$\Rightarrow$ B is an infinite set.

(iii) $\{x \in \mathbb{Z} : x$ is even and less than 10$\}$

Let C = $\{x \in \mathbb{Z}: x$ is even and $< 10\}$

$\Rightarrow$ C = $\{\ldots -4, -2, 0, 2, 4, 6, 8\}$. C is a infinite set.

(iv) $\{x \in \mathbb{R} : x$ is a rational number$\}$

Let D = $\{x \in \mathbb{R}: x$ is a rational number$\}$

$\Rightarrow$ D = {set of all rational number}

$\Rightarrow$ D is an infinite set.

(v) $\{x \in \mathbb{N} : x$ is a rational number$\}$

Let $\mathbb{N}$ = $\{x \in \mathbb{N}: x$ is a rational number$\}$

$\Rightarrow$ $\mathbb{N}$ = $\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \ldots \infty\right\}$

$\Rightarrow$ $\mathbb{N}$ is an infinite set.

4. By taking suitable sets A, B, C, verify the following results:

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$

(v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

Solution :

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Let A = $\{1, 2, 3\}$, B = $(4, 5, 6, 7\}$

C = $\{4,3,5,9\}$

and $\cup$ = $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

LHS = $A \times (B \cap C)$

= $A \times \{4, 5\}$ $[\because B \cap C = \{4,5\}]$

= $\{1, 2, 3\} \times \{4, 5\}$

$= \{(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)\}$ ...(1)

$A \times B = \{1, 2, 3\} \times \{4, 5, 6, 7\}$

$= \{(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)\}$

$A \times C = \{1, 2, 3\} \times \{3, 4, 5, 9\}$

$= \{(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)\}$

RHS $= (A \times B) \cap (A \times C)$

$= \{(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)\}$ ...(2)

From (1) and (2), LHS = RHS. Hence verified.

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$(B \cup C) = \{3, 4, 5, 6, 7, 9\}$

Now, $A \times (B \cup C) = \{1, 2, 3\} \times \{3, 4, 5, 6, 7, 9\}$

$= \{(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3, 9)\}$ ...(1)

Now $A \times B = \{1, 2, 3\} \times \{4, 5, 6, 7\}$

$= \{(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)\}$

$A \times C = \{1, 2, 3\} \times \{3, 4, 5, 9\}$

$= \{(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)\}$

RHS $(A \times B) \cup (A \times C)$

$= \{(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3,9)\}$ ...(2)

From (1) & (2), LHS = RHS Hence verified

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

$(A \times B) = \{(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)\}$

$(B \times A) = \{(4, 1) (4, 2) (4, 3) (5, 1) (5, 2) (5, 3) (6, 1) (6, 2) (6, 3) (7, 1) (7, 2) (7, 3)\}$

LHS $= (A \times B) \cap (B \times A) = \{\ \}$ ...(1)

$(A \cap B) = \{\ \}, (B \cap A) = \{\ \}$

$\therefore$ RHS $= (A \cap B) \times (B \cap A) = \{\ \}$ ...(2)

From (1) and (2), LHS = RHS

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$

$B - A = \{4, 5, 6, 7\}$

LHS $= C - (B - A) = \{3, 9\}$ ...(1)

$C \cap A = \{3\}$

$B' = \{1, 2, 3, 8, 9\}$

$C \cap B' = \{3, 9\}$

RHS $= (C \cap A) \cup (C \cap B')$

$= \{3, 9\}$ ...(2)

From (1) and (2), LHS = RHS

(v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

$B - A = \{4, 5, 6, 7\}$

$(B - A) \cap C = \{4, 5\}$ ...(1)

$B \cap C = \{4, 5\}$

$(B \cap C) - A = \{4, 5\}$ ...(2)

$C - A = \{4, 5, 9\}$

$B \cap (C - A) = 4, 5\}$ ...(3)

From (1), (2) and (3),

$(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$.

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

$B - A = \{4, 5, 6, 7\}$

$(B - A) \cup C = \{3, 4, 5, 6, 7, 9\}$ ...(1)

$B \cup C = \{3, 4, 5, 6, 7, 9\}$

$A - C = \{1, 2\}$

$(B \cup C) - (A - C) = \{3, 4, 5, 6, 7, 9\}$ ...(2)

From (1) and (2), $(B - A) \cup C = (B \cup C) - (A - C)$

Hence verified.

5. Justify the trueness of the statement "*An element of a set can never be a subset of itself*".

Solution : Let $P = \{a, b, c, d\}$.

Each and every element of the set P can be a subset of the set itself

Eg : $\{a\}, \{b\}, \{c\}, \{d\}$.

Hence, the given statement is not true.

6. If $n(p(A)) = 1024$, $n(A \cup B) = 15$ and $n(P(B)) = 32$, then find $n(A \cap B)$.

Solution : Given $n(P(A)) = 1024 = 2^{10}$ [$\therefore$ If $n(A) = n$, then $n(P(A)) = 2^n$]

$\Rightarrow n(A) = 10$

$n(P(B)) = 32 = 2^5$

$\Rightarrow n(B) = 5$.

We know that,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow 15 = 10 + 5 - n(A \cap B)$

$\Rightarrow n(A \cap B) = 0$.

7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$ then find $n(P(A \Delta B))$ **[Qy - 2018]**

Solution : We know that $n(A \cup B)$

$= n(A - B) + n(B - A) + n(A \cap B)$ if A and B are not disjoint.

$\Rightarrow n(A - B) + n(B - A) = n(A \cup B) - n(A \cap B)$

$\Rightarrow n(A \Delta B) = 10 - 3$

$\Rightarrow \therefore n(A \Delta B) = 7$

$\therefore n[P(A \Delta B)] = 2^7 = 128$.

8. For a set A, A × A contains 16 elements and two of its elements are (1, 3) and (0, 2). Find the elements of A.

Solution : Since A × A contains 16 elements, then A must have 4 elements

$\Rightarrow n(A) = 4$.

The elements of A × A are (1, 3) and (0, 2)

∴ The possibilities of elements of A are {0, 1, 2, 3}

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1)$ $(y, 2)$ $(z, 1)$ are in A × B, find A and B, where x, y, z are distinct elements. **[Hy - 2018]**

Solution : Given A × B $= \{(x, 1)\ (y, 2)\ (z, 1)\}$

Since $n(A) = 3$ and $n(B) = 2$,

A × B will have 6 elements.

The remaining elements of A × B will be $(x, 2)\ (y, 1)\ (z, 2)$

∴ A × B $= \{(x, 1)\ (y, 2)\ (z, 1)\ (x, 2)\ (y, 1)\ (z, 2)\}$

∴ A $= \{x, y, z\}$ and B = {1, 2}

10. If A × A has 16 elements, S = $\{(a, b) \in A \times A: a < b\}$; (–1, 2) and (0, 1) are two elements of S, then find the remaining elements of S. **[Qy - 2018]**

Solution : $n(A \times A) = 16 \Rightarrow n(A) = 4$.

Given S $= \{(a, b) \in A \times A: a < b\}$

∴ A = {–1,0, 1, 2}.

A × A = {(–1, –1) (–1, 0) (–1, 1) (–1, 2)(0, –1),(0,0) (0,1) (0,2) (1, –1) (1, 0)(1, 1) (1, 2) (2, –1)(2, 0) (2, 1) (2, 2)}

Now, S = {(–1, 0) (–1, 1) (–1, 2) (0, 1) (0, 2) (1,2)}

∴ The remaining elements of S are (–1, 0) (–1, 1) (0, 2) (1,2)

Exercise 1.2

1. Discuss the following relations for reflexivity, symmetricity and transitivity :

(i) The relation R defined on the set of all positive integers by "mRn if m divides n".

(ii) Let P denote the set of all straight lines in a plane. The relation R defined by "lRm if l is perpendicular to m".

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".

(v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".

Solution :

(i) The relation R defined on the set of all positive integers by "mRn" if m divides n".

Given relation is "mRn if m divides n".

Reflexivity : mRm since m divides m for all positive integers m.
∴ R is reflexive.

Symmetricity : mR$n \Rightarrow n$Rm.
m divides $n \Rightarrow$ 4 divides
2 ≠ 2 divides 4.
∴ R is not symmetric

Transitive : mRn and nR$p \Rightarrow m$Rp.
m divides n and n divides p then m divides p.
∴ R is transitive.

∴ R is reflexive, not symmetric and transitive.

(ii) Let P denote the set of all straight lines in a plane. The relation R defined by "lRm if l is perpendicular to m".

Let $l, m, n \in p$.

Reflexivity : We cannot say l is perpendicular to l itself.
∴ $l \not R\ l \Rightarrow$ R is not reflexive.

Symmetry : $lRm \Rightarrow mRl$
l is perpendicular to m = m is perpendicular to l
∴ R is symmetric

Transitive : lRm and $mRn \neq lRn$.
l is perpendicular to m and m is perpendicular to n.
⇒ l is not perpendicular to n.
∴ R is not transitive.
⇒ R is only symmetric.

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".

Given relation is "aRb if a is not a sister of b". and $a, b, c \in A$.

Reflexivity : aR$a \Rightarrow a$ is not a sister of a
∴ R is reflexive.

Symmetricity : aR$b \neq b$Ra
a is not a sister of b but may be a sister of a
∴ R is not symmetric.

Transitivity : aRb and bRc $\Rightarrow$ aRc
a is not a sister of b and b is not a sister of c.
but a may be a sister of c.
$\therefore$ R is not transitive.
$\therefore$ R is only reflexive.

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".

Given relation is aRb if a is not a sister of b.
Let $a, b, c \in$ A.

Reflexivity : aR$a \Rightarrow a$ is not a sister of a
$\therefore$ R is reflexive.

Symmetricity : aR$b \Rightarrow b$Ra
a is not a sister of $b \Rightarrow b$ is not a sister of a.
$\therefore$ R is symmetric.

Transitivity : aRb and bR$c \Rightarrow a$Rc
a is not a sister of b, b is not a sister of C [Eg : Mother is not a sister of daughter, daughter is not a sister of chithi, but mother is a sister of chithi.]
does not imply a is not a sister of c.
$\therefore$ R is not transitive.
$\therefore$ R is reflexive, symmetric and not transitive.

(v) On the set of natural numbers, the relation R is defined by "xRy if x + 2y = 1".

The relation R is defined by xRy if $x + 2y = 1$ for $x, y \in$ N.

Reflexivity : Let $x, y \in \mathbb{N}$
xR$x \Rightarrow x + 2x = 1$
$\Rightarrow 3x = 1 \quad \Rightarrow x = \frac{1}{3} \notin \mathbb{N}$
$\therefore$ R is not reflexive.

Symmetricity : xR$y \Rightarrow y$Rx for $x, y \in \mathbb{N}$
xR$y \Rightarrow x + 2y = 1$ which is not possible for any values of $x, y \in \mathbb{N}$
$\therefore$ R is not symmetric

Transitivity : xRy and yR$z \Rightarrow x$Rz.
xRy and yRz are not possible for any values of $x, y, z \in \mathbb{N}$
$\therefore$ R is not transitive.
$\therefore$ R is not reflexive, not symmetric and not transitive.

2. Let X = {a, b, c, d}, and R = {(a, a) (b, b) (a, c)}. Write down the minimum number of ordered pairs to be included to R to make it

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Solution : Given X = $\{a, b, c, d\}$ and R = $\{(a, a)\ (b, b)\ (a, c)\}$

(i) To make the relation R reflexive we must have (c, c) and $(d, d) \in$ R
$\therefore$ minimum number of ordered pairs to be included to R to make it reflexive is (c, c) and (d, d)

(ii) To make R symmetric, we must have $(c, a) \in$ R
$\therefore$ minimum number of ordered pairs to be included to R to make it symmetric is (c, a).

(iii) R is transitive.
$\therefore$ nothing need to be included.

(iv) Minimum number of ordered pairs to be included to make R equivalence is $(c, c)\ (d, d)\ (c, a)$.

3. Let A = {a, b, c}, and R = {(a, a) (b, b) (a, c)}. Write down the minimum number of ordered pairs to be included to R to make it

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Solution : (i) The ordered pairs (c, c) should be included to R to make it reflexive.
$\therefore$ minimum number of ordered pair is (c, c)

(ii) The ordered pairs (c,a) should be included to R to make it symmetric.
$\therefore$ minimum number of ordered pair is (c, a).

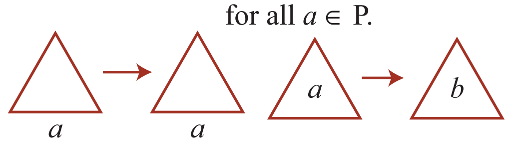
(iii) The relation is transitive.
$\therefore$ nothing needs to included.

(iv) The ordered pairs (c, c) and (c, a) should be included to R to make it equivalence.
$\therefore$ Minimum number of ordered pairs are (c, c) and (c, a).

4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.

Solution : Let P be the set of all triangles in a plane R is defined as aRb if a is similar to b.
Let $a, b, c \in$ P.

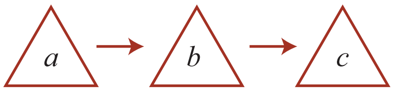
Reflexivity : aR$a \Rightarrow a$ is similar to a for all $a \in$ P.



$\therefore$ R is reflexive

Symmetricity : aR$b \Rightarrow b$Ra
a is similar to $b \Rightarrow b$ is similar to a for all $a, b \in$ P.

Transitivity : aRb, and bR$c \Rightarrow a$Rc.
a is similar to b and b is similar to c



$\Rightarrow a$ is similar to c.

Hence R is an equivalence relation.

5. On the set of natural number let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric

(iii) transitive (iv) equivalence.

Solution : Given relation is $2a + 3b = 30$ for all $a, b \in \mathbb{N}$.

$$2a + 3b = 30$$

$$\Rightarrow \quad 2a = 30 - 3b$$

$$\Rightarrow \quad a = \frac{30-3b}{2}$$

a	12	9	6	3
b	2	4	6	8

$\therefore$ The list of ordered pairs are (12, 2) (9, 4) (6, 6) (3, 8)

Reflexivity : $(12, 12) \notin$ R $\Rightarrow$ R is not reflexive.

Symmetricity : $(9, 4) \in$ R $\Rightarrow (4, 9) \notin \mathbb{R}$
$\therefore$ R is not symmetric

Transitivity : Clearly R is not transitive.
R is not an equivalence relation.

6. Prove that the relation "friendship" is not an equivalence relation on the set of all people in Chennai.

Solution : Let a, b, c are people in Chennai

Reflexivity : "a" is a friend of "a" $\Rightarrow a \not R\ a$.
$\Rightarrow$ R is not reflexive.

Symmetric : a is friend of $b \Rightarrow b$ is the friend of a.
$\therefore a$R$b \Rightarrow b$R$a \Rightarrow$ R is symmetric

Transitive : a is the friend of b and b is the friend of $c \Rightarrow a$ need not be the friend of c.
$\therefore a$R$b \Rightarrow b$R$c \neq a$R$c \Rightarrow$ R is not transitive

Hence, the relation "friendship" is not equivalent.

7. On the set of natural number let R be the relation defined by aRb if $a + b \leq 6$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric

(iii) transitive (iv) equivalence.

Solution : The relation is defined by aRb if $a + b \leq 6$ for all $a, b \in \mathbb{N}$.

$$a + b \leq 6 \Rightarrow a \leq 6 - b$$

a	5	4	3	2	1	1	1	1	1	2	2	2	3	4
b	1	2	3	4	5	1	2	3	4	1	2	3	1	1

$\therefore$ The list of ordered pairs are (5, 1) (4, 2) (3, 3) (2, 4) (1, 5), (1, 1), (1, 2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (4,1).

Reflexivity : R is not reflexive since $(5, 5) \notin \mathbb{R}$.

Symmetric : $(5, 1) \in \mathbb{R} \Rightarrow (1, 5) \in \mathbb{R}$
$(4, 2) \in \mathbb{R} \Rightarrow (2, 4) \in \mathbb{R}$
$\therefore$ R is symmetric

Transitivity : $(4, 2) \in \mathbb{R}$ and $(2, 4) \in \mathbb{R} \Rightarrow (4, 4) \notin$ R
$\therefore$ R is not transitive.

Hence, R is not an equivalence relation.

8. Let A = {a, b, c}. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?

Solution : Given A = {a, b, c}

(i) Let R = {(a, a) (b, b) (c,c)}
R is reflexive
R is symmetric and R is transitive $\Rightarrow$ R is an equivalence relation.
This is the equivalence relation of smallest cardinality on A.
$\therefore n$ (R) = 3

(ii) Let R = {(a, a) (a, b) (a,c) (b,a) (b, b) (b, c) (c, a) (c, b) (c,c)}
R is reflexive since (a, a) (b, b) and (c, c) $\in \mathbb{R}$
R is symmetric since $(a, b) \in$ R $\Rightarrow (b, a) \in \mathbb{R}$
$(b, c) \in \mathbb{R} \Rightarrow (c, b) \in \mathbb{R}$
$(c, a) \in \mathbb{R} \Rightarrow (a, c) \in \mathbb{R}$
R is also transitive since (a, b) $(b, c) \in \mathbb{R}$
$\Rightarrow (a, c) \in \mathbb{R}$
Hence R is are equivalence relation of largest cardinality on A.
$\therefore n$ (R) = 9

9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation.

Solution : As $m - m = 0$,
$m - m$ is divisible by 7 $\Rightarrow m$Rm
$\therefore$ R is reflexive. Let mRn.
Then $m - n = 7k$ for some integer k

Thus $n - m = 7(-k)$ and hence nRm

$\therefore$ R is symmetric.

Let mRn and nRp

$\Rightarrow \quad m - n = 7k$ and $n - p = 7l$ for some

$\Rightarrow \quad m = 7k + n$ and

$-p = 7l - n$ integers k and l

so $m - p = 7k + \cancel{n} + 7l - \cancel{n}$

$\Rightarrow \quad m - p = 7(k + l) \Rightarrow mRp$

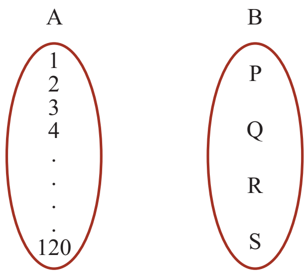
$\therefore$ R is transitive.

Thus, R is an equivalence relation.

Exercise 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "*x* related to *y* if the student *x* belongs to the section *y*". Is this relation a function? What can you say about the inverse relation? Explain your answer.

Solution : Given $n(A) = 120$, $n(B) = 4$



Let the four sections be P, Q, R and S.

xRy is the student x belongs to the section y.

This relation is a function since every student of set A will be mapped on to some section in B.

$\therefore f$ is a function from A $\to$ B.

The inverse relation is f^{-1}: B $\to$ A.

The inverse relation is not a function since one section will have more than one student.

2. Write the values of f at –4, 1, –2, 7, 0 if

$$f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \le -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \le x < 1 \\ x - x^2 & \text{if } 1 \le x < 7 \\ 0, & \text{otherwise} \end{cases}$$

Solution : Now $f(-4) = +4 + 4 = 8$

$[\because f(x) = -x + 4 \text{ when } x = -4]$

$f(1) = 1 - 1^2$

$[\because f(x) = x - x^2 \text{ when } x = 1]$

$f(1) = 0$

$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

$[\because f(x) = x^2 - x \text{ when } x = -2]$

$f(7) = 0 \quad [\because f(x) = 0 \text{ when } x = 7]$

$f(0) = 0^2 - 0 = 0.$

$[\because f(x) = x^2 - x \text{ when } x = 0]$

$\therefore f(-4) = 8, f(1) = 0,$

$f(-2) = 6, f(7) = 0$ and $f(0) = 1$

3. Write the values of f at –3, 5, 2, –1, 0 if

$$f(x) = \begin{cases} x^2 + x - 5 & \text{if } x \in (-\infty, 0) \\ x^2 + 3x - 2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2 - 3, & \text{Otherwise} \end{cases}$$

[First Mid - 2018]

Solution : $f(-3) = (-3)^2 - 3 - 5 = 9 - 3 - 5 = 9 - 8 = 1$

$[\because f(x) = x^2 + x - 5 \text{ when } x = -3]$

$f(5) = 5^2 + 3(5) - 2 = 25 + 15 - 2 = 38$

$[\because f(x) = x^2 + 3x - 2 \text{ when } x = 5]$

$f(2) = 2^2 - 3 = 4 - 3 = 1$

$[\because f(x) = x^2 - 3 \text{ when } x = 2]$

$f(-1) = (-1)^2 + (-1) - 5 = \cancel{1} - \cancel{1} - 5 = -5$

$[\because f(x) = x^2 + x - 5 \text{ when } x = -1]$

$f(0) = 0^2 - 3 = -3$

$[\because f(x) = x^2 - 3 \text{ when } x = 0]$

$\therefore f(-3) = 1, f(5) = 38,$

$f(2) = 1, f(-1) = -5, f(0) = -3$

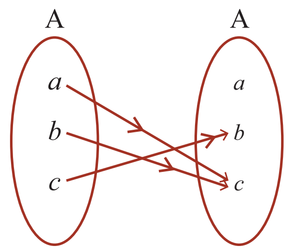
4. State whether the following relations are functions or not. If it is a function check for one-to-oneness and ontoness . If it is not a function state why?

(i) If A = $\{a, b, c\}$ and $f = \{(a, c)\ (b, c)\ (c, b)\}$: (f : A $\to$ A).

(ii) If X = $\{x, y, z\}$ and $f = \{(x, y)\ (x, z)\ (z, x)\}$: ($f$: X $\to$ X)

Solution : **(i) If A = $\{a, b, c\}$ and $f = \{(a, c)\ (b, c)\ (c, b)\}$ (f : A $\to$ A).**

Given f : A $\to$ A



This is a function. Since different elements of A does not have different images in A.

$\therefore f$ is not one-one.

Here

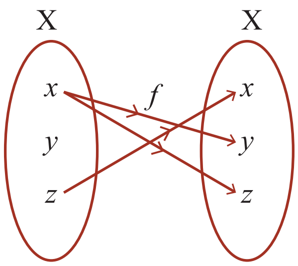
Co-domain $= \{a, b, c\}$

But Range $= \{b, c\}$

f is not onto since co-domain $\neq$ Range.

(ii) If X = $\{x, y, z\}$ and $f = \{(x, y)\ (x, z)\ (z, x)\}$: ($f$: X $\to$ X)

Given f : X $\to$ X



f is not a function since the element x have two images namely y and z.

5. Let A = {1, 2, 3, 4} and B = {a, b, c, d}. Give a function from A → B for each of the following :

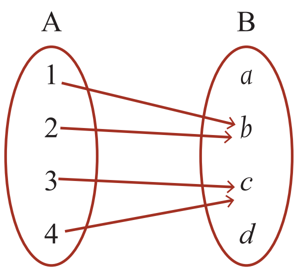
(i) neither one- to -one and nor onto.

(ii) not one-to-one but onto.

(iii) one-to-one but not onto.

(iv) one-to-one and onto.

Solution : **(i) neither one- to -one and nor onto.**



Let $f = \{(1, b)\ (2, b)\ (3, c)\ (4, c)\}$

Different elements in A does not have different images in B

$\therefore f$ is not one-one

Now, Co-domain = $\{a, b, c, d\}$,

Range = $\{b, c\}$ Co-domain ≠ range

$\therefore f$ is not onto. Hence f is neither one-one and nor onto.

(ii) not one-to-one but onto.

Given A = {1, 2, 3, 4}, and B = $\{a, b, c, d\}$

Let $f : A \to B$.

The function does not exist for not one-one but onto.

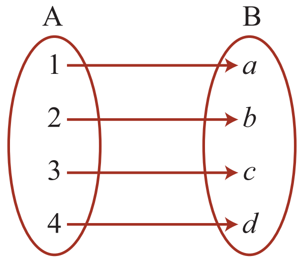
Since $f = A \to B$, f is onto $\Rightarrow f$ must be one one since $n(A) = n(B)$

(iii) one-to-one but not onto.

The function does not exist for one-to-one but not onto.

Since $f : A \to B$, f is one-one $\Rightarrow f$ must be onto $[\because n(A) = n(B)]$

(iv) one-to-one and onto.



Let f: A → B defined by

$f = \{(1, a)\ (2, b)\ (3, c)\ (4, d)\}$

Here different elements have different images

$\therefore f$ is one-to-one.

Also Co-domain = $\{a, b, c, d\}$ = Range.

$\therefore f$ is onto.

$\therefore f$ is one-to-one and onto.

6. Find the domain of $\dfrac{1}{1-2\sin x}$.

Solution : Let $f(x) = \dfrac{1}{1-2\sin x}$.

When the denominator is 0,

$$1 - 2\sin x = 0 \quad \Rightarrow 1 = 2\sin x$$

$$\Rightarrow \quad \sin x = \frac{1}{2} \Rightarrow \sin x = \sin\frac{\pi}{6}$$

$$\Rightarrow \quad x = n\pi + (-1)^n\frac{\pi}{6} \quad n \in \mathbb{Z}$$

$$[\because \sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n\alpha,\ n \in \mathbb{Z}]$$

Domain of $f(x)$ is $R - \left(n\pi + (-1)^n\dfrac{\pi}{6}\right), n \in \mathbb{Z}$

7. Find the largest possible domain of the real valued function $f(x) = \dfrac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$.

Solution : Given $f(x) = \dfrac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$.

When $x = 2, f(x) = 0$

When $x = -2, f(x) = 0$

For all the other values, we get negative value in the square root which is not possible.

$\therefore$ Domain $= \phi$

8. Find the range of the function $\dfrac{1}{2\cos x - 1}$.

[Govt. MQP - 2018]

Solution : Range of cosine function is $-1 \le \cos x \le 1$.

$\Rightarrow \quad -2 \le 2\cos x \le 2$ (Multiplied by 2)

$\Rightarrow \quad -2\ -1 \le 2\cos x - 1 \le 2 - 1$

$\Rightarrow \quad -3 \le 2\cos x - 1 \le 1$

$$\Rightarrow \frac{-1}{3} > \frac{1}{2\cos x - 1} > \frac{1}{1}$$

$$\Rightarrow \frac{-1}{3} > f(x) > 1$$

$\therefore$ Range of $f(x)$ is $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$

9. Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.

Solution : Given relation is $xy = -2$.

$$\Rightarrow x = -\frac{2}{y}$$

$$\text{Now } f(x_1) = f(x_2) \Rightarrow -\frac{2}{y_1} = -\frac{2}{y_2}$$

$$\Rightarrow \frac{1}{y_1} = \frac{1}{y_2} \Rightarrow y_1 = y_2$$

$\therefore f$ is a one-one function

The element $0 \in$ the domain will not have the image.

$\therefore$ Domain = R – {0} and Range = R – {0}.

10. If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, find $g \circ f$ and $f \circ g$.

Solution : Given $f(x) = |x| + x$

$$= \begin{cases} x + x = 2x & \text{if } x \geq 0 \\ -x + x = 0 & \text{if } x \leq 0 \end{cases}$$

$$g(x) = |x| - x$$

$$= \begin{cases} x - x = 0 & \text{if } x \geq 0 \\ -x - x = -2x & \text{if } x \leq 0 \end{cases}$$

$$\text{Now, } fog(x) = f(g(x))$$

$$= \begin{cases} f(0) & \text{if } x \geq 0 \\ f(-2x) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow fog(x) = \begin{cases} 2 \times 0 = 0 & \text{if } x \geq 0 \\ 2(-2x) = -4x & \text{if } x < 0 \end{cases}$$

$$\text{and } gof(x) = g(f(x)) = \begin{cases} g(2x) & \text{if } x \geq 0 \\ g(0) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow gof(x) = \begin{cases} 0 \text{ if } x \geq 0 \\ 0 \text{ if } x < 0 \end{cases}$$

$$\Rightarrow gof(x) = 0 \text{ for all } x \in \mathbb{R}.$$

11. If f, g, h are real valued functions defined on $\mathbb{R}$, then prove that $(f + g) oh = f o h + g o h$. What can you say about $f o (g + h)$? Justify your answer.

Solution :

(i) Since f, g, h are functions from $\mathbb{R} \to \mathbb{R}$,
$(f + g) oh: \mathbb{R} \to \mathbb{R}$ and $foh + goh: \mathbb{R} \to \mathbb{R}$.
For any $x \in \mathbb{R}$,

$$[(f + g) oh](x) = (f + g)(h(x))$$
$$= f(h(x)) + g(h(x))$$
$$= foh(x) + goh(x)$$
$$\therefore (f + g)oh = foh + goh$$

(ii) Also $fo(g + h) = f[(g + h)(x)]$ for any $x \in \mathbb{R}$

$$= f[g(x) + h(x)]$$
$$= f(g(x) + f(h(x))$$
$$= fog(x) + foh(x).$$
$$\therefore fo(g + h) = fog(x) + foh(x).$$

12. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, prove that f is a bijection and find its inverse.

[Govt.MQP & Qy- 2018]

Solution : Let $y = 3x - 5$.

$$\Rightarrow y + 5 = 3x \Rightarrow \frac{y+5}{3} = x.$$

$$\text{Let } g(y) = \frac{y+5}{3}.$$

$$gof(x) = g(f(x)) = g(3x - 5)$$

$$= \frac{3x - \cancel{5} + \cancel{5}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

$$\text{Also } fog(y) = f(g(y)) = f\left(\frac{y+5}{3}\right)$$

$$= 3\left(\frac{y+5}{3}\right) - 5 = y + 5 - 5 = y.$$

Thus $gof(x) = \text{I}x$ and $fog(x) = \text{I}y$.

Where I is identify function.

This implies that f and g are bijections and inverses to each other.

Hence f is a bijection and $f^{-1}(y) = \frac{y+5}{3}$.

Replacing y by x we get, $f^{-1}(x) = \frac{x+5}{3}$

13. The weight of the muscles of a man is a function of his body weight x and can be expressed as W(x) = 0.35x. Determine the domain of this function.

Solution : Given W(x) = 0.35x

(Note that x is positive real numbers)

W(0) = 0, W(1) = 0.35,

W(2) = 7, W(∞) = ∞

Domain (0, ∞) Range (0, ∞)

14. The distance of an object falling is a function of time t and can be expressed as $s(t) = -16t^2$. Graph the function and determine if it is one-to-one.

Solution : Given $s(t) = -16t^2$

Now, $s(t_1) = s(t_2)$

$\Rightarrow \quad -16t_1^2 = -16t_2^2$

$\Rightarrow \quad t_1^{\,2} = t_2^2$

$\Rightarrow \quad \pm t_1 = \pm t_2$

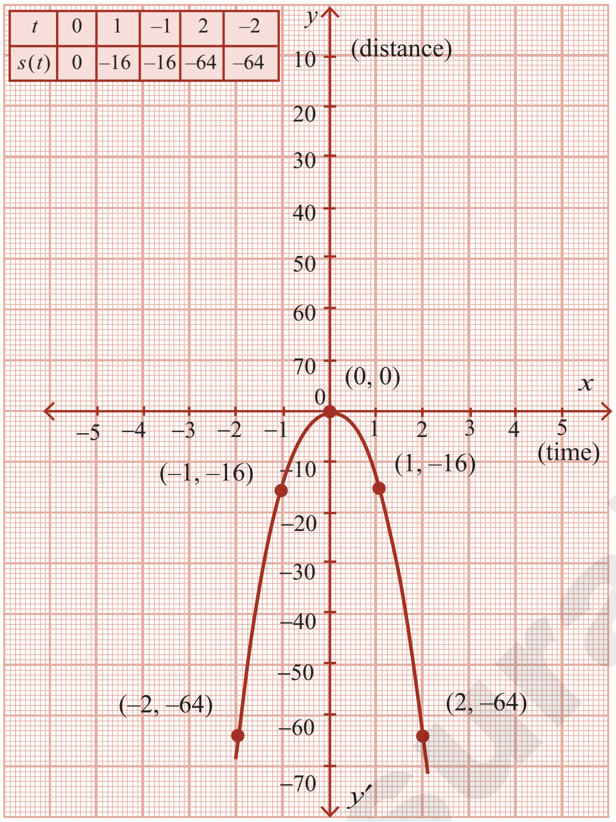
Since $s(t_1) = s(t_2) \neq t_1 = t_2$,

the function $s(t)$ is one-one.

Graph of $s(t) = -16t^2$

Let X - axis represents the time and Y - axis represents the distance.

t	0	1	–1	2	–2
$s(t)$	0	–16	–16	–64	–64



15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; C(m) = 0.4 m + 50 and S(m) = 0.03 m. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.

Solution : Given cost function and fuel surcharge function are as follows:

$c(m) = 0.4\,m + 50$

and $s(m) = 0.03\,m$.

$\therefore$ Total cost of a ticket $= c(m) + s(m)$

$\therefore f(x) = 0.4\,m + 50 + 0.03\,m$

$= 0.43\,m + 50$

Given m = 1600 miles

Airfare for flying 1600 miles = 0.43(1600) +50

= ₹738

16. A salesperson whose annual earnings can be represented by the function A(x) = 30,000 + 0.04x, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function S(x) = 25,000 + 0.05x. Find (A + S) (x) and determine the total family income if they each sell ₹1,50,00,000 worth of merchandise.

Solution : Given A(x) = 30,000 + 0.04 x

S(x) = 25,000 + 0.05 x.

$\therefore$ (A + S) (x) = 30,000 + 0.04x + 25,000 + 0.05x

= 55,000 + 0.09

Given x = ₹1,50,00,000

Then Family income is = 55,000 + 0.09 (1,50,00,000)

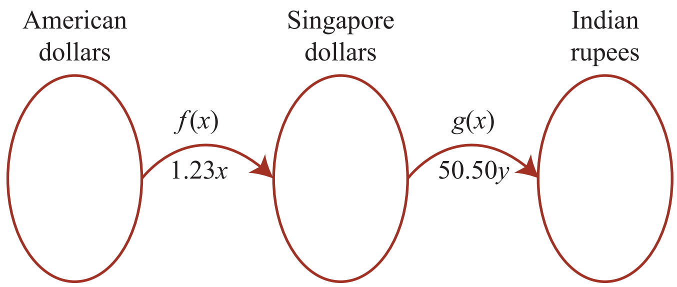
= 55,000 + 13,50,000

= 14,05,000.

Hence total family income = ₹ 14,05,000.

17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x)$ = 1.23x, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is g(y) = 50.50y, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.

Solution : Given $f(x) = 1.23x$ where x represents the number of American dollars and $g(y) = 50.50y$ where y represents the number of Singapore dollars.



To convert American dollars to Indian rupees, we have to find out go $f(x)$

$\therefore$ go $f(x) = g(f(x)) = g(1.23x)$

$= 50.50[1.23x] = 62.115x$

$\therefore$The function for exchange rate of American dollars in terms of Indian rupee is $gof(x) = 62.115x$.

18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimate that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function D(x) = 200 – x. Express his day revenue total cost and profit on this meal as a function of x.

Solution : Number of customers = $200 - x$

Cost of one meal = ₹100

Total cost = $100\,(200 - x)$

Revenue on one meal = x

Total revenue = $x\,(200 - x)$

Profit = Revenue – Cost

= ₹ $x(200 - x) - 100\,(200 - x)$

= ₹ $(200 - x)\,(x - 100)$

19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \frac{5x}{9} - \frac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.

Solution : Let $f(x) = \frac{5x - 160}{9}$

Given $y = \frac{5x}{9} - \frac{160}{9} \Rightarrow y = \frac{5x - 160}{9}$

Then $9y = 5x - 160$

$\Rightarrow 5x = 9y + 160 \Rightarrow x = \frac{9y + 160}{5}$

Let $g(y) = \frac{9y + 160}{5}$.

Now $gof(x) = g[f(x)] = g\left(\frac{5x - 160}{9}\right)$

$$= \frac{\cancel{9}\left(\frac{5x - 160}{\cancel{9}}\right) + 160}{5}$$

$$= \frac{5x - \cancel{160} + \cancel{160}}{5} = \frac{\cancel{5}x}{\cancel{5}} = x$$

and $fog(y) = f(g(y)) = f\left(\frac{9y + 160}{5}\right)$

$$= \frac{\cancel{5}\left(\frac{9y + 160}{\cancel{5}}\right) - 160}{9}$$

$$= \frac{9y + \cancel{160} - \cancel{160}}{9} = y$$

Thus $gof = I_x$ and $fog = I_y$.

This implies that f and g are bijections and inverses to each other.

$$f^{-1}(y) = \frac{9y + 160}{5}$$

Replacing y by x, we get $f^{-1}(x) = \frac{9x + 160}{5} = \frac{9x}{5} + 32$

20. A simple cipher takes a number and codes it, using the function $f(x) = 3x - 4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawing the lines).

Solution : Given $f(x) = 3x - 4$

Let $y = 3x - 4 \Rightarrow y + 4 = 3x$

$\Rightarrow x = \frac{y + 4}{3}$

Let $g(y) = \frac{y + 4}{3}$.

Now $gof(x) = g(f(x)) = g(3x - 4)$

$$= \frac{3x - \cancel{4} + \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

and $fog(y) = f(g(y)) = f\left(\frac{y + 4}{3}\right)$

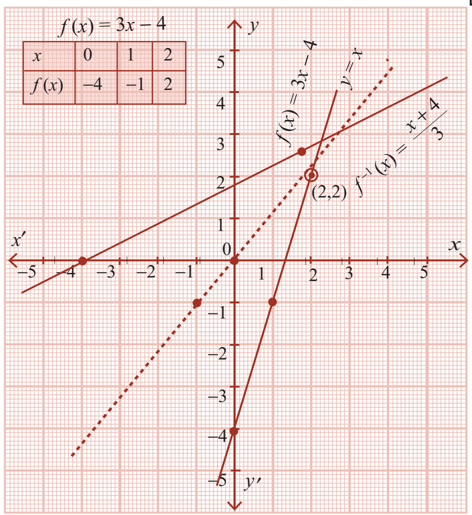
$$= \cancel{3}\left(\frac{y + 4}{\cancel{3}}\right) - 4 = y + \cancel{4} - \cancel{4} = y$$

Thus, $gof(x) = I_x$ and $fog(y) = I_y$.

This implies that f and g are bijections and inverses to each other.

Hence f is bijection and $f^{-1}(y) = \frac{y + 4}{3}$

Replacing y by x, we get $f^{-1}(x) = \boxed{\frac{x + 4}{3}}$.



Hence, the graph of $y = f^{-1}(x)$ is the reflection of the graph of f in $y = x$

Exercise 1.4

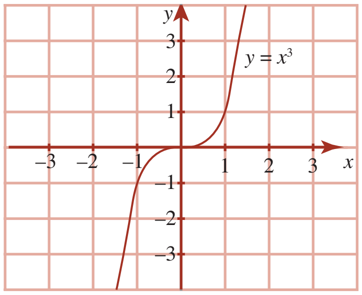
1. For the curve $y = x^3$ given in figure draw,

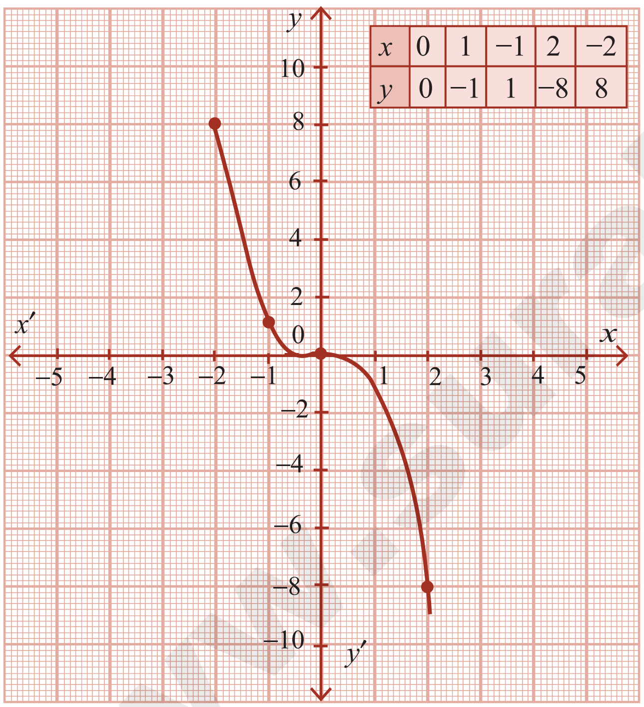
(i) $y = -x^3$ (ii) $y = x^3 + 1$

(iii) $y = x^3 - 1$ (iv) $y = (x + 1)^3$

with the same scale.

Solution : (i) $y = -x^3$.



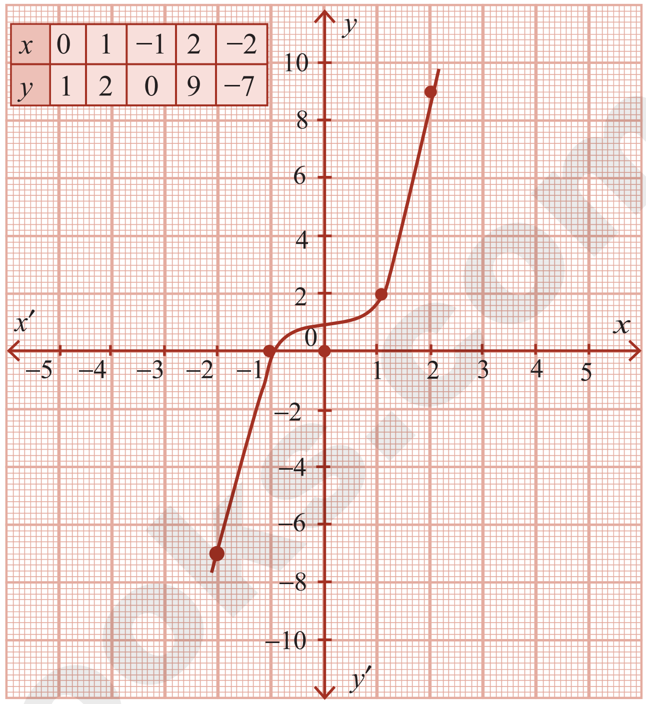


x	0	1	−1	2	−2
y	0	−1	1	−8	8

Let $f(x) = x^3$

Since $y = -f(x)$, this is the reflection of the graph of f about the x-axis

(ii) $y = x^3 + 1$

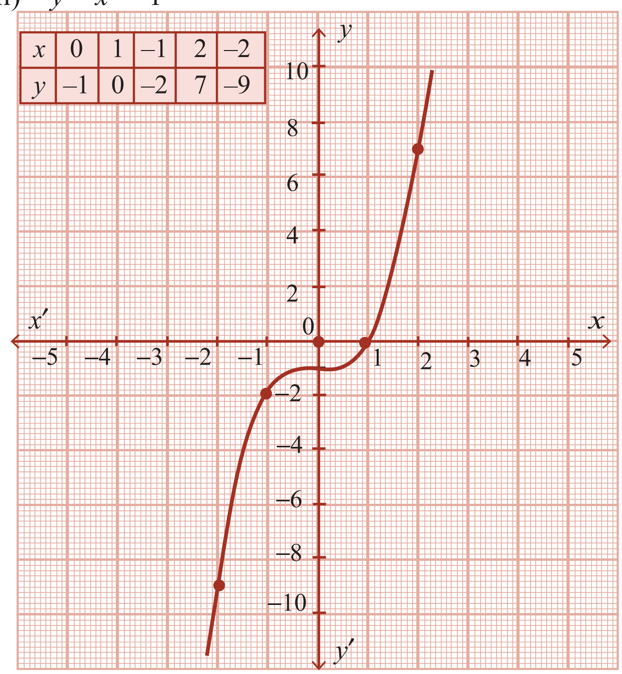


x	0	1	−1	2	−2
y	1	2	0	9	−7

Let $f(x) = x^3$

Since $y = f(x) + 1$, this is the graph of $f(x)$ shifts to the upward for one unit

(iii) $y = x^3 - 1$

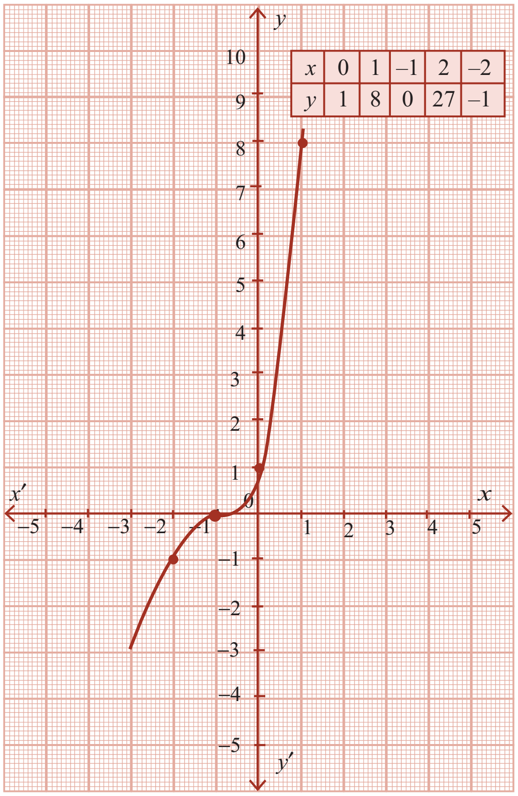


x	0	1	−1	2	−2
y	−1	0	−2	7	−9

Let $f(x) = x^3$

Since $y = f(x) - 1$, this is the graph of $f(x)$ shifts to the downward for one unit.

(iv) $y = (x + 1)^3$ **[Govt.MQP - 2018]**



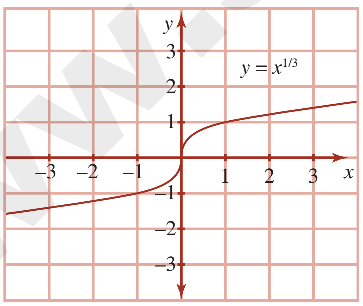
Let $f(x) = x^3$

$y = (x + 1)^3$, causes the graph of $f(x)$ shifts to the left for one unit.

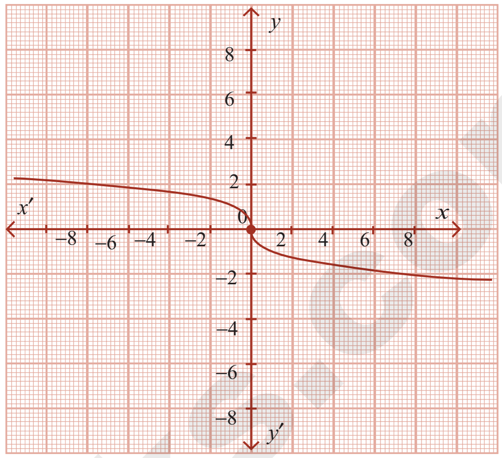
2. For the curve, $y = x^{\frac{1}{3}}$ given in figure draw.

(i) $y = -x^{\left(\frac{1}{3}\right)}$ **(ii)** $y = x^{\left(\frac{1}{3}\right)} + 1$

(iii) $y = x^{\left(\frac{1}{3}\right)} - 1$ **(iv)** $(x+1)^{\left(\frac{1}{3}\right)}$

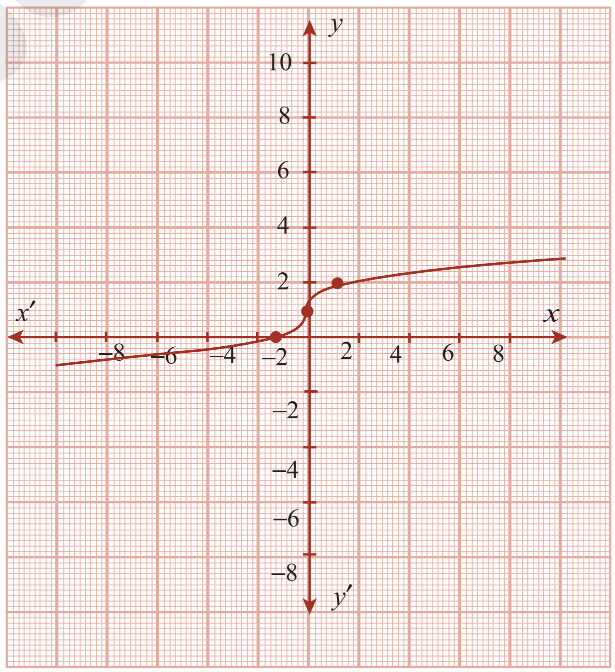


Solution : (i) $y = -x^{\frac{1}{3}}$



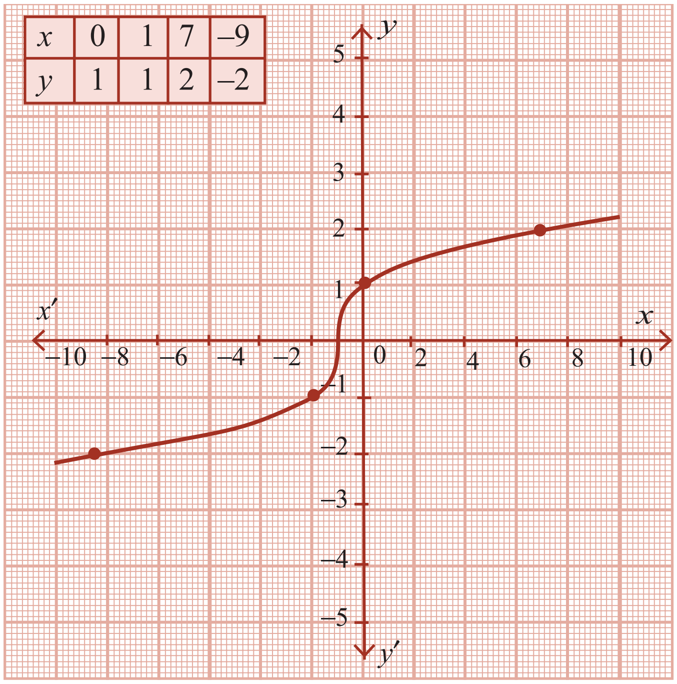
Then $y = -x^{\frac{1}{3}}$ is the reflection of the graph of $y = x^{\frac{1}{3}}$ about the x-axis.

(ii) $y = x^{\frac{1}{3}} + 1$.



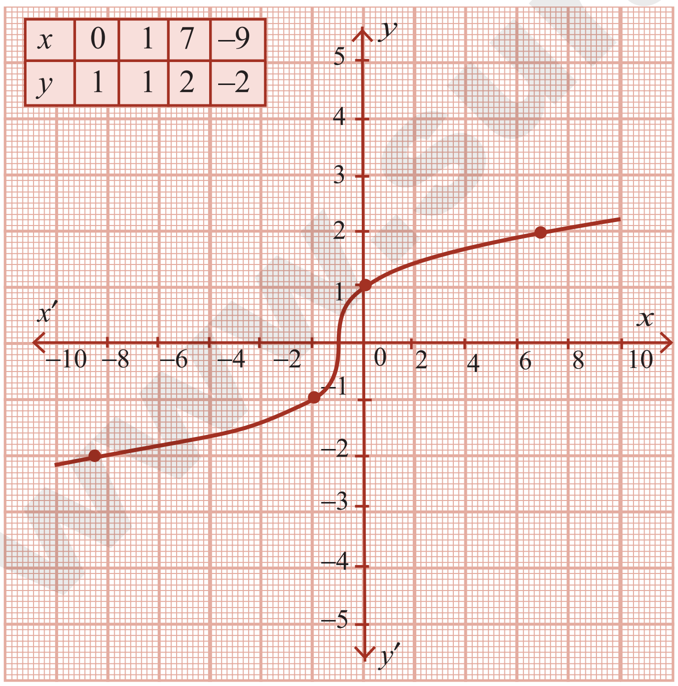
Then $y = x^{\frac{1}{3}} + 1$ is the x graph of $y = x^{\frac{1}{3}}$ shifts to the upward for one unit.

(iii) $y = x^{\frac{1}{3}} - 1$.



Then $y = x^{\frac{1}{3}} - 1$ is the graph of $x^{\frac{1}{3}}$ shifts to the downward for one unit.

(iv) $y = (x+1)^{\frac{1}{3}}$

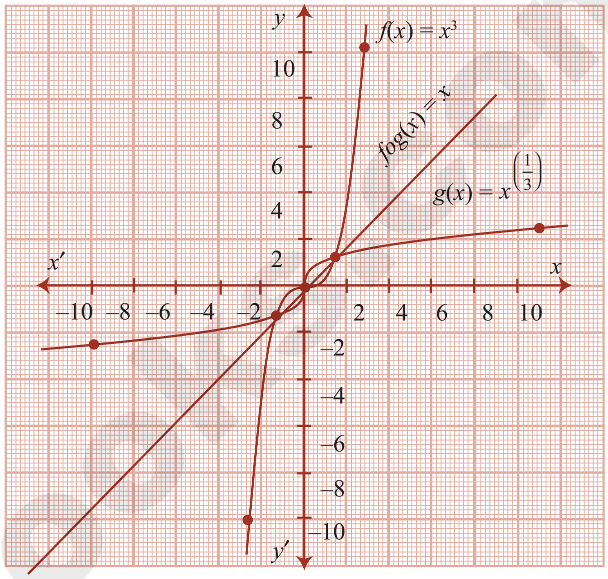


$y = (x+1)^{\frac{1}{3}}$, it causes the graph of $x^{\frac{1}{3}}$, shifts to the left for one unit.

3. Graph the functions $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ on the same co-ordinate plane. Find fog and graph it on the plane as well. Explain your results.

Solution : Given functions are $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$.

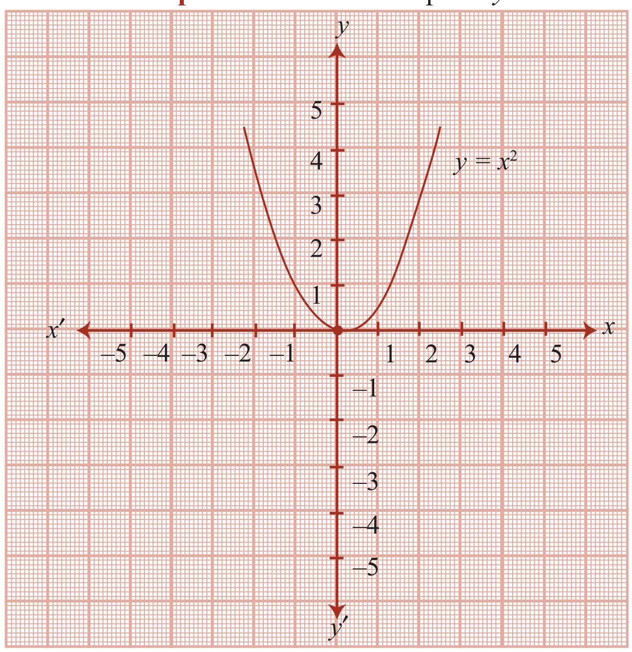
Now, $fog(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = \left(x^3\right)^{\frac{1}{3}} = x$



Since fog$(x) = x$ is symmetric about the line $y = x$, $g(x)$ is the inverse of $f(x)$ $\therefore g(x) = f^{-1}(x)$.

4. Write the steps to obtain the graph of the function $y = 3(x-1)^2 + 5$ from the graph $y = x^2$.

Solution : **Step 1 :** Draw the Graph of $y = x^2$



Step 2 :

The graph of $y = (x-1)^2$, shifts to the right for 1 unit.

Step 3 :

The graph of $y = 3(x-1)^2$, compresses towards the Y - axis that is moves away from the X-axis since the multiplying factor is 3 which is greater than 1.

Step 4 :

The graph of $y = 3(x-1)^2 + 5$, causes the shift to the upward for 5 units.

5. From the curve $y = \sin x$, graph the functions.

(i) $y = \sin(-x)$ (ii) $y = -\sin(-x)$,

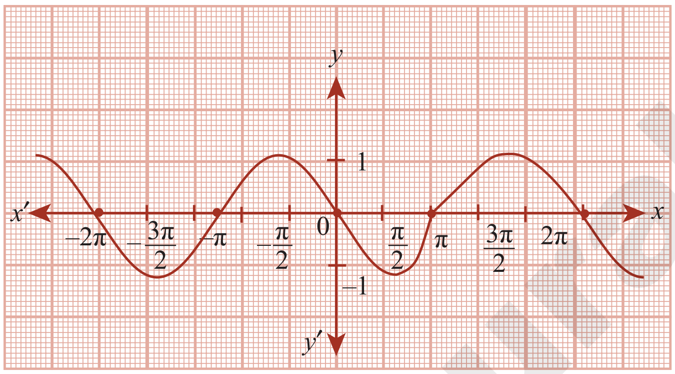
(iii) $y = \sin\left(\frac{\pi}{2} + x\right)$ which is cos x.

(iv) $y = \sin\left(\frac{\pi}{2} - x\right)$ which is also cos x.

(refer trigonometry)

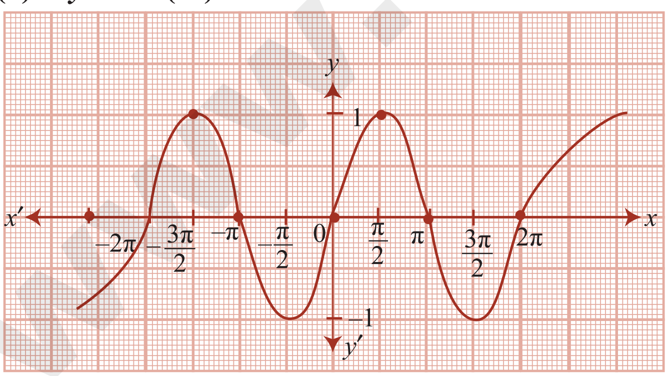
Solution :

(i) $y = \sin(-x)$



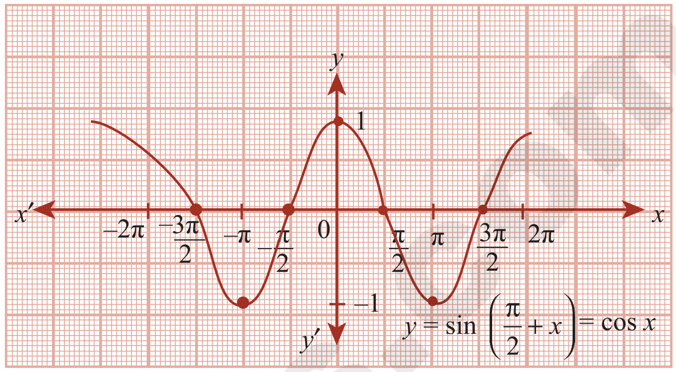
Then $y = \sin(-x)$ is the reflection of the graph of sin x, about y-axis.

(ii) $y = -\sin(-x)$



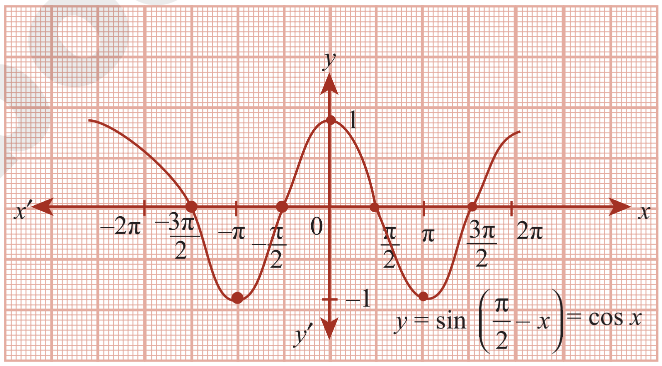
$y = -\sin(-x)$ is the reflection of $y = \sin(-x)$ which is same as $y = \sin x$.

(iii) $y = \sin\left(\frac{\pi}{2} + x\right)$



Then $y = \sin\left(\frac{\pi}{2} + x\right)$ it causes the shift to the left for $\frac{\pi}{2}$ units.

(iv) $y = \sin\left(\frac{\pi}{2} - x\right)$



Then $y = \sin\left(\frac{\pi}{2} - x\right)$ causes the shift to the right for $\frac{\pi}{2}$ unit to the sin($-x$) curve.

6. From the curve $y = x$, draw

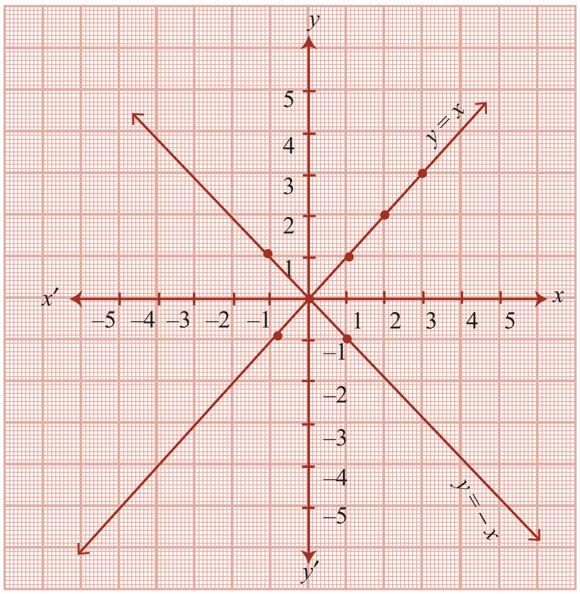
(i) $y = -x$ (ii) $y = 2x$

(iii) $y = x + 1$ (iv) $y = \frac{1}{2}x + 1$

(v) $2x + y + 3 = 0$.

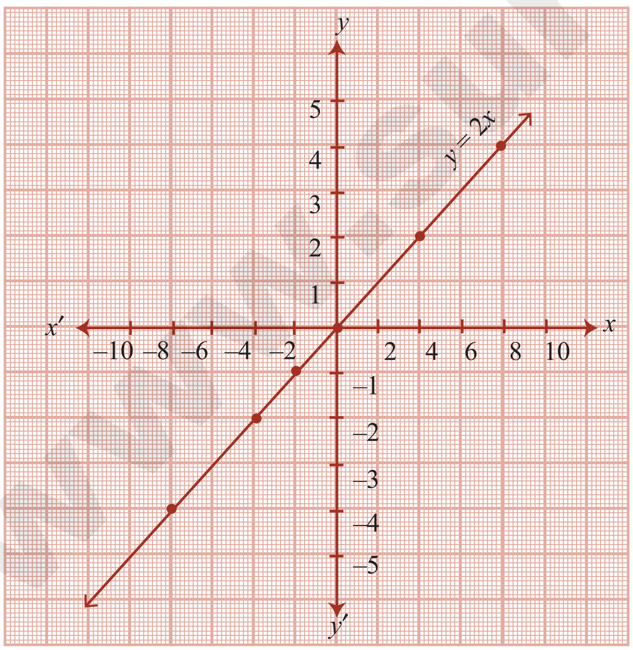
Solution : Graph of $y = x$ and

(i) $y = -x$



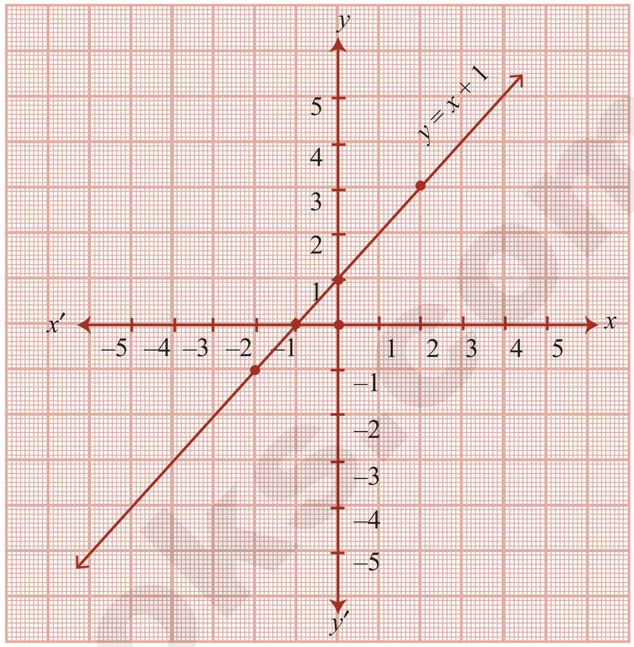
Graph of $y = -x$ is the reflection of the graph of $y = x$ about the X - axis.

(ii) $y = x + 1$



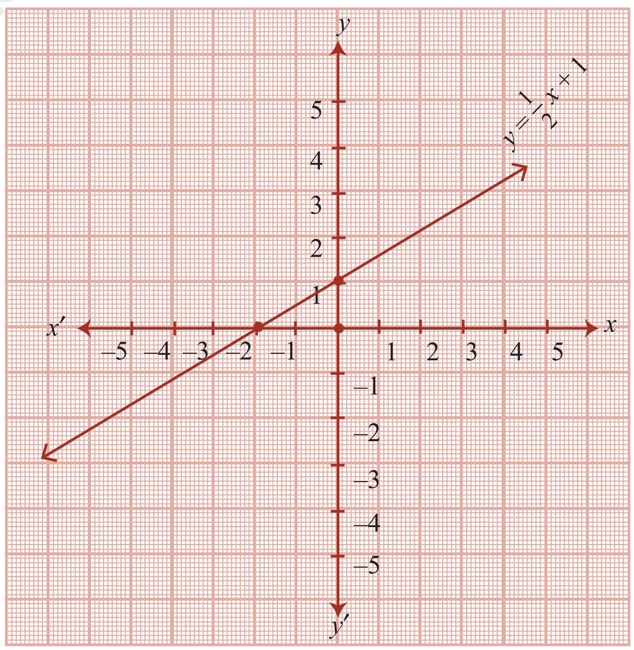
The graph of $y = 2x$ compresses towards the Y-axis that is moves away from the X-axis since the multiplying factor is 2, which is greater than 1.

(iii) $y = x + 1$



The graph of $y = x + 1$, causes the shift to the upward for one unit.

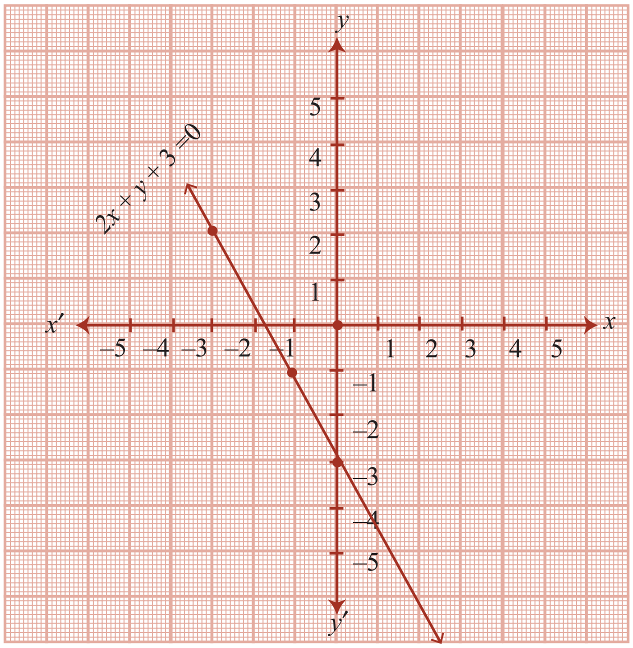
(iv) $y = \frac{1}{2}x + 1$



The graph of $y = \frac{1}{2}x + 1$, stretches towards the X-axis since the multiplying factor is $\frac{1}{2}$ which is less than one and shifts to the upward for one unit.

(v) $2x + y + 3 = 0.$

$\Rightarrow y = -2x - 3$



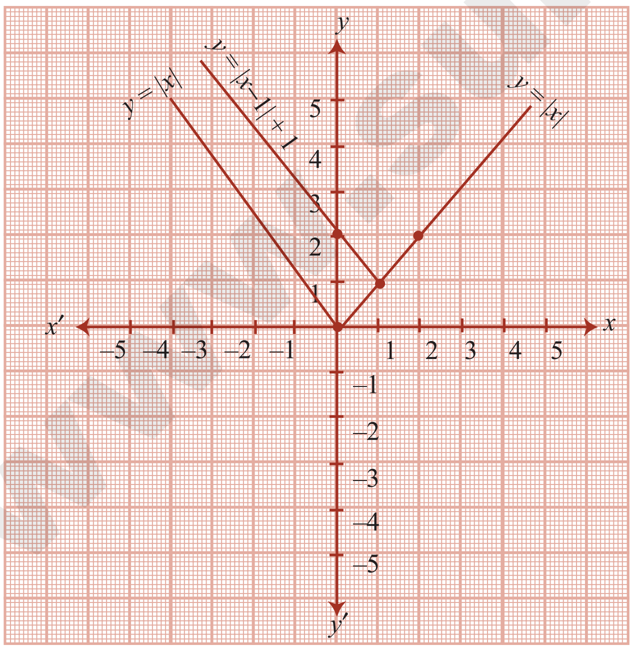
The graph of $y = -2x - 3$, stretches towards the X-axis since the multiplying factor is -2 which is less than one and causes the shifts to the downward for 3 units.

7. From the curve $y = |x|$, draw

(i) $y = |x - 1| + 1$ (ii) $y = |x + 1| - 1$

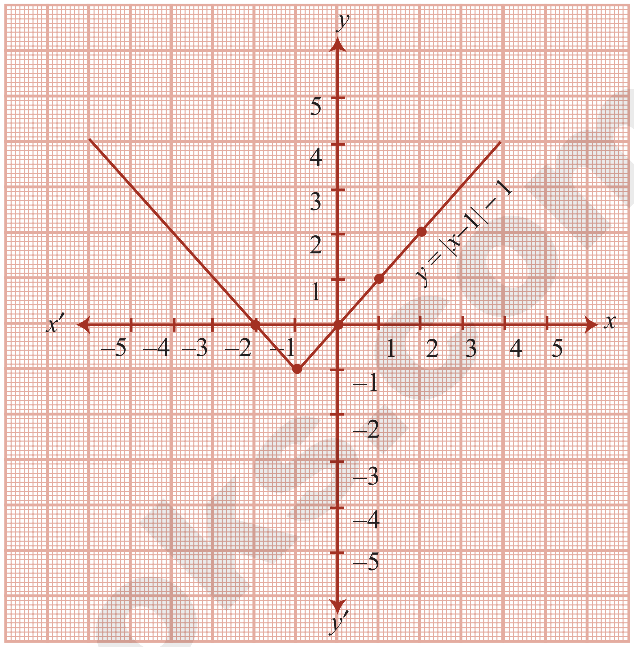
(iii) $y = |x + 2| - 3.$

Solution : **(i) $y = |x - 1| + 1$**



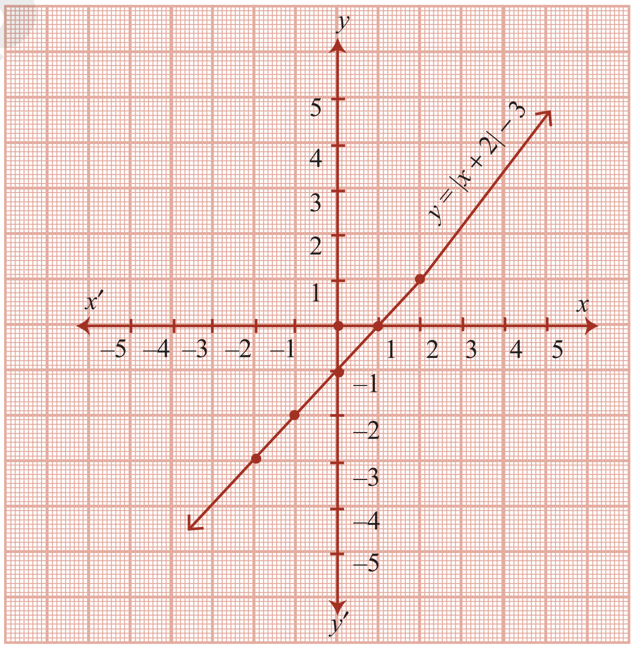
The graph of $y = |x - 1| + 1$, shifts to the right for one unit and causes the shift to the upward for one unit.

(ii) $y = |x + 1| - 1$



The graph of $y = |x + 1| - 1$, shifts to the left for one unit and causes the shift to the downward for one unit.

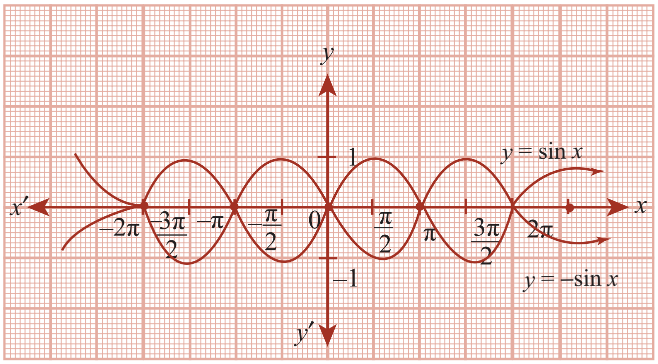
(iii) $y = |x + 2| - 3$



The graph of $y = |x + 2| - 3$, shifts to the left for 2 units and causes the shift to the downward for 3 units.

8. From the curve $y = \sin x$, draw $y = \sin |x|$ (Hint: $\sin(-x) = -\sin x$.)

Solution : $y = \sin |x|$



$$\text{We know } |x| = \begin{cases} x \text{ if } & x \geq 0 \\ -x \text{ if } & x < 0 \end{cases}$$

$$\therefore \sin |x| = \sin x \text{ if } x \geq 0$$

$$\text{and } \sin |x| = \sin(-x) = -\sin x \text{ if } x < 0.$$

The graph of $y = \sin(-x) = -\sin x$ is the reflection of the graph of $\sin x$ about Y - axis.

Exercise 1.5

Choose the correct or the most suitable answer.

1. If A = $\{(x, y) : y = e^x, x \in \mathbb{R}\}$ and B = $\{(x, y) : y = e^{-x}, x \in \mathbb{R}\}$ then $n(A \cap B)$ is **[First Mid - 2018]**

(1) Infinity (2) 0 (3) 1 (4) 2

Hint : $n(A \cap B) = 1$ **[Ans :** (3) 1**]**

2. If A = $\{(x, y) : y = \sin x, x \in \mathbb{R}\}$ and B = $\{(x, y) : y = \cos x, x \in \mathbb{R}\}$ then $A \cap B$ contains

(1) no element **[Govt. MQP - 2018]**
(2) infinitely many elements
(3) only one element
(4) cannot be determined.

[Ans : (2) infinitely many elements**]**

3. The relation R defined on a set A = {0,–1, 1, 2} by xRy if $|x^2 + y^2| \leq 2$, then which one of the following is true?

(1) R = {(0, 0), (0,–1), (0, 1), (–1, 0), (–1, 1), (1, 2), (1, 0)}
(2) R^{-1} = {(0, 0), (0,–1), (0, 1), (–1, 0), (1, 0)}
(3) Domain of R is {0,–1, 1, 2}
(4) Range of R is {0,–1, 1}

Hint : Since $|x^2 + y^2| < 2$, x, y must be 0 or 1

[Ans : (4) Range of R is {0, –1, 1}**]**

4. If $f(x) = |x - 2| + |x + 2|$, $x \in \mathbb{R}$, then

$$(1)\ f(x) = \begin{cases} -2x & \text{if} & x \in (-\infty, -2] \\ 4 & \text{if} & x \in (-2, 2] \\ 2x & \textit{if} & x \in (2, \infty) \end{cases}$$

$$(2)\ f(x) = \begin{cases} 2x & \text{if} & x \in (-\infty, -2] \\ 4 & \text{if} & x \in (-2, 2] \\ -2x & \text{if} & x \in (2, \infty) \end{cases}$$

$$(3)\ f(x) = \begin{cases} -2x & \text{if} & x \in (-\infty, -2] \\ -4 & \text{if} & x \in (-2, 2] \\ 2x & \text{if} & x \in (2, \infty) \end{cases}$$

$$(4)\ f(x) = \begin{cases} -2x & \text{if} & x \in (-\infty, -2] \\ 2 & \text{if} & x \in (-2, 2] \\ 2x & \text{if} & x \in (2, \infty) \end{cases}$$

Hint : Let $x \in (-\infty, -2)$,

$$\text{let } x = -3 \text{ then}$$
$$f(x) = |-5| + |1| = 6 = -2x$$
$$x \in (-2, 2), \text{ let } x = 0 \text{ then}$$
$$f(x) = |0 - 2| + |0 + 2| = 4$$
$$x \in (2, \infty), \text{ let } x = 4 \text{ then}$$
$$f(x) = |2| + |6| = 8 = 2x$$

[Ans : (1) $f(x) = \begin{cases} -2x & \text{if} & x \in (-\infty, -2] \\ 4 & \text{if} & x \in (-2, 2] \\ 2x & \textit{if} & x \in (2, \infty) \end{cases}$ **]**

5. Let $\mathbb{R}$ be the set of all real numbers. Consider the following subsets of the plane $\mathbb{R} \times \mathbb{R}$: S = $\{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$ and T = $\{(x, y) : x - y$ is an integer$\}$. Then which of the following is true?

(1) T is an equivalence relation but S is not an equivalence relation.
(2) Neither S nor T is an equivalence relation
(3) Both S and T are equivalence relation
(4) S is an equivalence relation but T is not an equivalence relation.

Hint : $x - y$ is an integer $\Rightarrow x$Ry

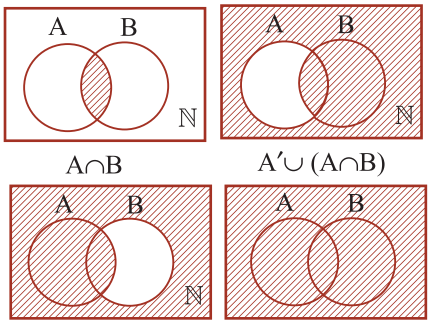
(i) $x - x = 0$ is an integer. $\therefore x$Rx reflexive
(ii) $(x - y)$ is an integer $\Rightarrow y - x$ is also an integer $\Rightarrow$ symmetric
(iii) If $(x - y)$ is an integer $\Rightarrow y - z$ is an integer by adding $x - z$ is also an integer.
$\therefore$ T is equivalence.
(iv) $y = x + 1 \Rightarrow x$Sx is not true. S is not an equivalence relation.
$\therefore$ T is an equivalence relation but S is not.

[Ans : (1) T is an equivalence relation but S is not an equivalence relation**]**

6. Let A and B be subsets of the universal set $\mathbb{N}$, the set of natural numbers. Then $A' \cup [(A \cap B) \cup B']$ is **[First Mid - 2018]**

(1) A (2) A′ (3) B (4) $\mathbb{N}$

Hint :



$A \cap B$ $\quad$ $A' \cup (A \cap B)$

$(A \cap B) \cup B'$ $\quad$ $A' \cup (A \cap B) \cup B'$

[Ans : (4) $\mathbb{N}$]

7. The number of students who take both the subjects Mathematics and Chemistry is 70. This represents 10% of the enrollment in Mathematics and 14% of the enrollment in Chemistry. The number of students take at least one of these two subjects, is

(1) 1120 (2) 1130
(3) 1100 (4) insufficient data

Hint : $M \cap C = 70$

Which is 10% of M and 14% of C

$M = 700$

$C = 500$

$M \cup C = 700 + 500 - 70 = 1130$

[Ans : (2) 1130]

8. If $n((A \times B) \cap (A \times C)) = 8$ and $n(B \cap C) = 2$, then $n(A)$ is

(1) 6 (2) 4 (3) 8 (4) 16

Hint : $(A \times B) \cap (A \times C) = A \times (B \cap C)$

$n[(A \times B) \cap (A \times C)] = 8$

$n(B \cap C) = 2$

$n(A) = 4$ **[Ans : (2) 4]**

9. If $n(A) = 2$ and $n(B \cup C) = 3$, then $n[(A \times B) \cup (A \times C)]$ is **[Qy - 2018]**

(1) 2^3 (2) 3^2 (3) 6 (4) 5

Hint : $n[(A \times B) \cup (A \times C)] = n(A) \times n(B \cup C)$

$= 2 \times 3 = 6$ **[Ans : (3) 6]**

10. If two sets A and B have 17 elements in common, then the number of elements common to the set $A \times B$ and $B \times A$ is

(1) 2^{17} (2) 17^2
(3) 34 (4) insufficient data

Hint : Let $A = \{1, 2, 3, 4\}$

$B = \{5, 2, 3, 6\}$

A and B have two elements in common

Number of elements common to $A \times B$ and $B \times A = 2 \times 2 = 2^2$

Similarly here we have 17^2 elements common

[Ans : (2) 17^2]

11. For non-empty sets A and B, if $A \subset B$ then $(A \times B) \cap (B \times A)$ is equal to **[Hy- 2018]**

(1) $A \cap B$ (2) $A \times A$
(3) $B \times B$ (4) none of these.

Hint : Let $A = (a, b)$ $B = (a, b, c)$

$A \times B = \{(a, a), (a, b) (a, c) (b, a) (b, b) (b, c)\}$

$B \times A = \{(a, a), (a, b) (b, a) (b, b) (c, a) (c, b)\}$

$(A \times B) \cap (B \times A) = \{(a, a), (a, b) (b, a) (b, b)\}$

$= A \times A$ **[Ans : (2) $A \times A$]**

12. The number of relations on a set containing 3 elements is **[Govt. MQP & First Mid - 2018]**

(1) 9 (2) 81 (3) 512 (4) 1024

Hint : Let $S = \{a, b, c\}$

$n(S) = 3 \Rightarrow n(S \times S) = 9$

Number of relations is $n\{P(S \times S)\} = 2^9 = 512$

[Ans : (3) 512]

13. Let R be the universal relation on a set X with more than one element. Then R is

(1) not reflexive (2) not symmetric
(3) transitive (4) none of the above

Hint : Let $X = \{a, b, c\}$

Then R = Universal relation

$= \{(a, a), (a, b) (a, c) (b, a) (b, b) (b, c) (c, a) (c, b) (c, c)\}$

It is transitive **[Ans : (3) transitive]**

14. Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3), (2, 1), (3, 1), (1, 4), (4, 1)\}$. Then R is **[First Mid - 2018]**

(1) reflexive (2) symmetric
(3) transitive (4) equivalence

Hint : $(4,4) \in R$ not reflexive

Symmetric can be easily checked $\Rightarrow$ if aRb then bRc. **[Ans : (2) symmetric]**

15. The range of the function $\dfrac{1}{1 - 2\sin x}$ is

(1) $(-\infty, -1) \cup \left(\dfrac{1}{3}, \infty\right)$ (2) $\left(-1, \dfrac{1}{3}\right)$

(3) $\left[-1, \dfrac{1}{3}\right]$ (4) $(-\infty, -1] \cup \left[\dfrac{1}{3}, \infty\right)$

Hint : $-1 \le \sin x \le 1$

$-2 \le 2\sin x \le 2$

$$2 \geq -2\sin x \geq -2$$
$$\text{(or) } -2 \leq -2\sin x \leq 2$$
Adding, 1, $1-2 \leq 1-2\sin x \leq 1+2$
$$-1 \leq 1-2\sin x \leq 3$$
$$-1 \geq \frac{1}{1-2\sin x} \geq \frac{1}{3}$$
$$\frac{1}{3} \leq \frac{1}{1-2\sin x} \leq -1 \quad \text{Range is } (-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$$

[Ans : (4) $(-\infty, -1] \cup \left[\frac{1}{3}, \infty\right)$**]**

16. The range of the function $f(x) = |\lfloor x \rfloor - x|, x \in \mathbb{R}$ is

(1) $[0, 1]$ (2) $[0, \infty)$ (3) $[0, 1)$ (4) $(0, 1)$

Hint :
$$f(x) = |\lfloor x \rfloor - x|$$
$$f(x) = \lfloor x \rfloor - x$$
$$f(0) = 0 - 0 = 0$$
$$f(6.5) = 6 - 6.5 = |-.5| = .5$$
$$f(-7.2) = 8 - 7.2 = .8$$
Range is $[0, 1)$ **[Ans : (3) $[0,1)$]**

17. The rule $f(x) = x^2$ is a bijection if the domain and the co-domain are given by

(1) $\mathbb{R}, \mathbb{R}$ (2) $\mathbb{R}, (0, \infty)$
(3) $(0, \infty); \mathbb{R}$ (4) $[0, \infty); [0, \infty)$

Hint : The domain is $(0, \infty)$
The codomain is also $(0, \infty)$ **[Ans : (4) $[0, \infty); [0, \infty)$]**

18. The number of constant functions from a set containing m elements to a set containing n elements is

(1) mn (2) m (3) n (4) $m + n$

Hint : By definition it follows **[Ans : (3) n]**

19. The function $f : [0, 2\pi] \to [-1, 1]$ defined by $f(x) = \sin x$ is **[Govt. MQP - 2018]**

(1) one-to-one (2) onto
(3) bijection (4) cannot be defined

Hint : It is onto not one-one
$$\text{since } \sin 30^\circ = \frac{1}{2}$$
$$\sin 150^\circ = \frac{1}{2}$$
[Ans : (2) onto]

20. If the function $f : [-3, 3] \to S$ defined by $f(x) = x^2$ is onto, then S is

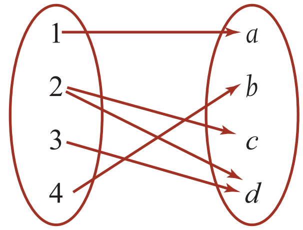
(1) $[-9, 9]$ (2) $\mathbb{R}$ (3) $[-3, 3]$ (4) $[0, 9]$

Hint : $f(0) = 0, f(-3) = 9$ and $f(3) = 9$ **[Ans : (4) $[0, 9]$]**

21. Let X = {1, 2, 3, 4}, Y = {a, b, c, d} and f = {(1, a), (4, b), (2, c), (3, d), (2, d)}. Then f is

(1) an one-to-one function
(2) an onto function
(3) a function which is not one-to-one
(4) not a function

Hint : It is not a function since it has two images.



[Ans : (4) not a function]

22. The inverse of $f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } 1 \leq x \leq 4 \\ 8\sqrt{x} & \text{if } x > 4 \end{cases}$ **is**

(1) $f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \leq x \leq 16 \\ \frac{x^2}{64} & \text{if } x > 16 \end{cases}$

(2) $f^{-1}(x) = \begin{cases} -x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \leq x \leq 16 \\ \frac{x^2}{64} & \text{if } x > 16 \end{cases}$

(3) $f^{-1}(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \leq x \leq 16 \\ \frac{x^2}{64} & \text{if } x > 16 \end{cases}$

(4) $f^{-1}(x) = \begin{cases} 2x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \leq x \leq 16 \\ \frac{x^2}{8} & \text{if } x > 16 \end{cases}$

Hint : (1) Let $y = x$ then $x = y \Rightarrow f^{-1}(x) = x$
Let $y = x^2$ then
$y = \sqrt{x} \Rightarrow f^{-1}(x) = \sqrt{x}$
Let $y = 8\sqrt{x}$ then $\frac{y^2}{64} = x \Rightarrow f^{-1}(x) = \frac{x^2}{64}$

[Ans : (1) $f^{-1}(x) = \begin{cases} x & \text{if } x < 1 \\ \sqrt{x} & \text{if } 1 \leq x \leq 16 \\ \frac{x^2}{64} & \text{if } x > 16 \end{cases}$**]**

23. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - |x|$. Then the range of f is **[Govt. MQP; Qy & Hy - 2018]**

(1) $\mathbb{R}$ (2) $(1, \infty)$
(3) $(-1, \infty)$ (4) $(-\infty, 1]$

Hint : f : R $\to$ R defined by

$$f(x) = 1 - |x|$$

The range is $(-\infty, 1]$, $f(-\infty) = -\infty$

$$f(0) = 1$$
$$f(\infty) = -\infty$$

[Ans : (4) $(-\infty, 1]$]

24. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \sin x + \cos x$ is

(1) an odd function
(2) neither an odd function nor an even function
(3) an even function
(4) both odd function and even function.

Hint :
$$f(x) = \sin x + \cos x$$
$$f(-x) = \sin(-x) + \cos(-x) = -\sin x + \cos x$$
$$-f(-x) = \sin x - \cos x$$

$f(x)$ is neither odd function nor even function.

[Ans : (2) neither an odd function nor an even function]

25. The function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = \dfrac{(x^2 + \cos x)(1 + x^4)}{(x - \sin x)(2x - x^3)} + e^{-|x|}$ is

(1) an odd function
(2) neither an odd function nor an even function
(3) an even function
(4) both odd function and even function.

Hint :
$$f(x) = \frac{(x^2 + \cos x)}{(x - \sin x)(2x - x^3)} + e^{-|x|}$$
$$f(-x) = \frac{(-x^2) + \cos(-x)}{[-x - \sin(-x)][-2x - (-x)^3]} + e^{-|x|}$$
$$= \frac{x^2 + \cos x}{(x - \sin x)(2x - x^3)} + e^{-|x|} = f(x)$$

Here $f(x)$ is even function.

[Ans : (3) an even function]

ADDITIONAL PROBLEMS

SECTION - A

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER.

1. Let f: R → R be defined as $f(x) = x^4$. Choose the correct answer. [Qy - 2018]

(1) f is one - one onto (2) f is onto
(3) f is one - one but not onto
(4) f is neither one - one nor onto

[Ans : (4) f is neither one - one nor onto]

2. Let f: R → R to given by $f(x) = (3 - x^3)^{\frac{1}{3}}$. then $fof(x)$ is [Qy - 2018]

(1) $x^{\frac{1}{a}}$ (2) x^a (3) x (4) $3 - x^a$

Hint :
$$fof(x) = f\left[(3 - x^3)^{\frac{1}{3}}\right]$$
$$= \left(3 - [3 - x^3]\right)^{\frac{1}{3}} = x$$

[Ans : (3) x]

3. Let A = {–2, –1, 0, 1, 2} and f : A → Z be given by $f(x) = x^2 - 2x - 3$ then preimage of 5 is [First Mid - 2018]

(1) –2 (2) –1 (3) 0 (4) 1

Hint :
$$f(-2) = (-2)^2 - 2(-2) - 3 = 4 + 4 - 3 = 5$$

[Ans : (1) –2]

4. If A = {$(x, y)/y = e^x$, $x \in [0, \infty)$} and B = {$(x, y)/y = \sin x$, $x \in [0, \infty)$} then $n(A \cap B)$ is [March - 2019]

(1) ∞ (2) 1 (3) ϕ (4) 0

Hint : $n(A \cap B) = 0$ **[Ans : (4) 0]**

5. If : R → R is defined by $f(x) = |x| - 5$, then the range of f is: [March - 2019]

(1) $(-\infty, -5)$ (2) $(-\infty, 5)$
(3) $[-5, \infty)$ (4) $(-5, \infty)$

Hint :
$$0 \le |x| < \infty, x \in \mathbb{R}$$
$$0 - 5 \le |x| - 5 < \infty$$
$$-5 \le |x| - 5 < \infty$$

[Ans : (3) $[-5, \infty)$]

6. Which one of the following is a finite set?

(1) $\{x : x \in \mathbb{Z}, x < 5\}$
(2) $\{x : x \in \mathbb{W}, x \ge 5\}$
(3) $\{x : x \in \mathbb{N}, x > 10\}$
(4) $\{x : x$ is an even prime number$\}$

Hint : $\{x : x$ is an even prime number$\} = \{2\}$

[Ans : (4) $\{x : x$ is an even prime number$\}$]

7. If A ⊆ B, then A\B is

(1) B (2) A (3) ∅ (4) $\frac{B}{A}$

Hint : If A ⊆ B, then every element of A is element of B, So $\frac{A}{B}$ is ∅. **[Ans : (3) ∅]**

8. Given A = {5, 6, 7, 8}. Which one of the following is incorrect?

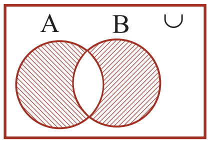
(1) ∅ ⊆ A (2) A ⊆ A
(3) {7, 8, 9} ⊆ A (4) {5} ⊂ A

Hint : 9 ∉ A, So {7, 8, 9} ⊈ A **[Ans : (3) {7, 8, 9} ⊆ A]**

9. The shaded region in the adjoining diagram represents.

(1) A\B (2) B\A (3) A Δ B (4) A′

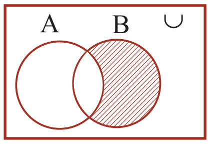
Hint : (A – B)∪ (B – A) = A Δ B **[Ans : (3) A Δ B]**



10. The shaded region in the adjoining diagram represents.

(1) A\B (2) A′ (3) B′ (4) B\A

[Ans : (4) B\A]



11. Let R be a relation on the set ℕ given by ℝ = {(*a*, *b*): *a* = *b* – 2, *b* > 6}. Then

(1) (2, 4) ∈ R (2) (3, 8) ∈ R
(3) (6, 8) ∈ R (4) (8, 7) ∈ R

Hint : 6 = 8 – 2 ⇒ 6 = 6 **[Ans : (3) (6, 8) ∈ R]**

12. If A = {1, 2, 3}, B = {1, 4, 6, 9} and R is a relation from A to B defined by "*x* is greater than *y*". The range of R is

(1) {1, 4, 6, 9} (2) {4, 6, 9}
(3) {1} (4) none of these

Hint : {(2, 1), (3, 1)} **[Ans : (3) {1}]**

13. For real numbers *x* and *y*, define *x*R*y* if $x - y + \sqrt{2}$ is an irrational number. Then the relation R is

(1) reflexive (2) symmetric
(3) transitive (4) none of these

Hint : $x \text{ R } x \Rightarrow \not{x} - \not{x} + \sqrt{2} = \sqrt{2}$, irrational R is reflexive. **[Ans : (1) reflexive]**

14. Let R be the relation over the set of all straight lines in a plane such that $l_1 R l_2 \Leftrightarrow l_1 \perp l_2$. Then R is

(1) symmetric (2) reflexive
(3) transitive (4) an equivalence relation

Hint : $l_1 \perp l_2 \Rightarrow l_2 \perp l_1$
⇒ R is symmetric **[Ans : (1) symmetric]**

15. Which of the following is not an equivalence relation on *z*?

(1) $a\text{R}b \Leftrightarrow a + b$ is an even integer
(2) $a\text{R}b \Leftrightarrow a - b$ is an even integer
(3) $a\text{R}b \Leftrightarrow a < b$
(4) $a\text{R}b \Leftrightarrow a = b$

Hint : *a* not less than *a*.
∴ *a*R*b* *ab* is not an equivalence relation.
[Ans : (3) $a\text{R}b \Leftrightarrow a < b$]

16. Which of the following functions from *z* to itself are bijections (one-one and onto)?

(1) $f(x) = x^3$ (2) $f(x) = x + 2$
(3) $f(x) = 2x + 1$ (4) $f(x) = x^2 + x$

[Ans : (2) $f(x) = x + 2$]

17. Let *f*: Z → Z be given by $f(x) = \begin{cases} \frac{x}{2} \text{ if } x \text{ is even} \\ 0 \text{ if } x \text{ is odd} \end{cases}$. Then *f* is

(1) one-one but not onto
(2) onto but not one-one
(3) one-one and onto
(4) neither one-one nor onto

Hint : *f* (3) = *f* (5) = 0. Hence *f* is not one-one.
[Ans : (2) onto but not one-one]

18. If *f*: R → R is given by *f*(*x*) = 3*x* – 5, then $f^{-1}(x)$ is

(1) $\frac{1}{3x-5}$ (2) $\frac{x+5}{3}$
(3) does not exist since f is not one-one
(4) does not exists since *f* is not onto

Hint :
$$y = 3x - 5$$
$$\Rightarrow \frac{y+5}{3} = x$$
$$\Rightarrow g(y) = \frac{y+5}{3}$$
$$\Rightarrow g(x) = \frac{x+5}{3}$$
[Ans : (2) $\frac{x+5}{3}$]

19. If *f*(*x*) = 2*x* – 3 and *g*(*x*) = $x^2 + x - 2$ then gof(*x*) is

(1) $2(2x^2 - 5x + 2)$ (2) $(2x^2 - 5x - 2)$
(3) $2(2x^2 + 5x + 2)$ (4) $2x^2 + 5x - 2$

Hint :
$$gof(x) = (2x-3)^2 + 2x - 3 - 2$$
$$= 4x^2 + 9 - 12x + 2x - 3 - 2$$
$$= 1\,(2x^2 - 5x + 2)$$
[Ans : (1) $2(2x^2 - 5x + 2)$]

20. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + \sqrt{x^2}$ is

(1) injective (2) surjective
(3) bijective (4) none of these

[Ans : (4) none of these**]**

Hint : f is neither one - one nor onto.

21. Choose the correct statement.

(1) One-to-one function have inverse
(2) Onto function have inverse
(3) bijection function have inverse
(4) many - to - one function hae inverse

[Ans : (3) bijection function have inverse**]**

22. Match List - I with List II

	List I		List II
i.	{(1, 1), (2,2), (3,3)(1,2)}	(a)	equivalence
ii.	{(1,2), (2,1), (2,3), (3,2)}	(b)	transitive
iii.	{(1,2), (2,3), (1,3)}	(c)	Symmetric
iv.	{(1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (1,3)}	(d)	reflexive

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	c	d	b	a
(2)	d	c	b	a
(3)	b	a	d	c
(4)	b	a	b	c

[Ans : (2) i – d ii – c iii – b iv – a**]**

SECTION - B

1. If n (A∩B) = 3 and n (A∪B) = 10 then find n [P(AΔB)] **[Qy - 2018]**

Solution :

n (AΔB) = n (A∪B) – n (A∩B)
n (AΔB) = 10 – 3 = 7
n (P (AΔB)) = $2^7 = 128$

2. In the set Z of integers, define mRn if $m - n$ is a multiple of 12. Prove that R is an equivalence relation. **[Govt. MQP & Qy - 2018]**

Solution : As $m - m = 0$ and $0 = 0 \times 12$, hence mRm proving that R is reflexive.

Let mRn. Then $m - n = 12k$ for some integer k; thus

$n - m = 12(-k)$ and hence nRm.

This shows that R is symmetric.

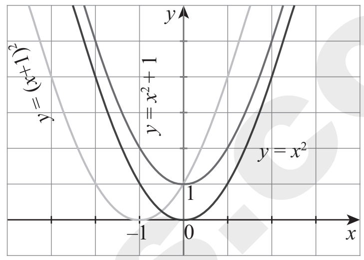
Let mRn and nRp; then

$m - n = 12k$
and $n - p = 12l$ for some integers k and l.
So $m - p = 12(k + l)$ and hence mRp.

This shows that R is transitive.

3. Draw the curves of (i) $y = x^2 + 1$ (ii) $y = (x + 1)^2$ by using the graph of curve $y = x$. **[Hy - 2018]**

Solution :



$f(x) = x^{2+1}$ causes the graph of the function $f(x) = x^2$ shifts to the upward for one unit.

$f(x) = (x + 1)^2$ causes the graph of the function $f(x) = x^2$ shifts to the left for one unit.

4. Find the number of subsets of A if A = {X : X = 4n + 1, 2 ≤ n ≤ 5, n∈ ℕ} **[First Mid - 2018]**

Solution : Clearly A = $\{x : x = 4n + 1, n = 2, 3, 4, 5\}$

$n = 2 \Rightarrow 4(2) + 1 = 8 + 1 = 9$
$n = 3 \Rightarrow 4(3) + 1 = 12 + 1 = 13$
$n = 4 \Rightarrow 4(4) + 1 = 16 + 1 = 17$
$n = 5 \Rightarrow 4(5) + 1 = 20 + 1 = 21$
A = $\{x : x = 9, 13, 17, 21\}$
Hence n(A) = 4. This implies that
n(P(A)) = $2^4 = 16$.

5. Let f = {(1, 4) (2, 5) (3, 5)} and g = {(4, 1) (5, 2) (6, 4)} find gof. Can you find fog? **[First Mid - 2018]**

Solution : Clearly, gof = {(1, 1), (2, 2), (3, 2)}

But fog is not defined because the range of
g = {1, 2, 4} is not contained in the domain of
f = {1, 2, 3}.

6. Define one to one function? **[First Mid - 2018]**

Solution : A function is said to be one-to-one if each element of the range is associated with exactly one element of the domain. i.e. two different elements in the domain(A) have different images in the co-domain(B).

7. If A = {1, 2, 3, 4} and B = {3, 4, 5, 6}, find n ((A∪B) × (A∩B) × (AΔB)). **[Govt. MQP- 2018]**

Solution : We have n (A∪ B) = 6,
n (A∩ B) = 2 and
n(A Δ B) = 4

Volume - II

MATHEMATICS

11th Standard

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07 Matrices and Determinants

MUST KNOW DEFINITIONS

Matrix : A matrix is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].

Order of Matrix : If a matrix A has m rows and n columns then the order or size of the matrix A is defined to be $m \times n$.

Column Matrix : A matrix having only one column is called a column matrix.

Row matrix : A matrix having only one row is called a row matrix.

Square matrix : A matrix in which number of rows is equal to the number of columns, is called a square matrix.

Diagonal matrix : A square matrix $A = [a_{ij}]_{m \times n}$ is called a diagonal matrix. If $a_{ij} = 0$ whenever $i \neq j$

Scalar matrix : A diagonal matrix whose entries along the principal diagonal are equal is called a scalar matrix.

Unit matrix : A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a unit matrix.

Triangular matrix : A square matrix which is either upper triangular or lower triangular is called a triangular matrix.

Singular and Non - Singular Matrix : A square matrix A is said to be singular if $|A| = 0$. A square matrix A is said to be non-singular if $|A| \neq 0$.

Properties of Determinants :

1. The value of the determinant remains unchanged if its rows and columns are interchanged.
2. If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
3. If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
4. If each element of a row (or column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.
5. If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.
6. The value of the determinant remain same if we apply the operation. $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Minor of an element

- The concept of determinant can be extended to the case of square matrix or order n, $n \geq 4$. Let $A = [a_{ij}]_{m\times n}$, $n \geq 4$.
- If we delete the i^{th} row and j^{th} column from the matrix of $A = [a_{ij}]_{n\times m}$, we obtain a determinant of order $(n - 1)$, which is called the minor of the element a_{ij}.

Adjoint

- Adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$ where A_{ij} is the co-factor of the element a_{ij}.

Solving linear equations by Gaussian Elimination method

- Gaussian elimination is a method of solving a linear system by bringing the augmented matrix to an upper triangular form.

FORMULAE TO REMEMBER

- $kA = [a_{ij}]_{m\times n}$ $[ka_{ij}]_{m\times n}$ where k is a scalar.
- $-A = (-1)A$, $A - B = A + (-1)B$
- $A + B = B + A$, (Commutative property for addition)
- $(A + B) + C = A + (B + C)$, (Associative property for addition)
- $k(A + B) = kA + kB$ where A, B are of same order, k is a constant.
- $(k + l)\, A = kA + lA$ where k and l are constants.
- $A\,(BC) = (AB)\,C$, $A(B + C) = AB + AC$, $(A + B)\,C = AC + BC$. (Distributive law)
- If $A = (a_{ij})_{m\times n}$, then $A^T = (a_{ji})_{n\times m}$
- Elementary operations of a matrix are as follows
  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ (ii) $R_i \to kR_i$ or $C_i \to kC_j$ (iii) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$
- Evaluation of determinant $A = [a_{11}]_{1\times 1} = |A| = a_{11}$
- Evaluation of determinant $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
- Evaluation of determinant $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $|A| = a_1 \begin{vmatrix} b_2 & c_3 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$
- If $A = [a_{ij}]_{3\times 3}$, then $|k \cdot A| = k^3\,|A|$.
- $A\,(adj\,A) = (adj\,A)\,A = |A|.\,I$ where A is a square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- **Transpose of a matrix:** $(A^T)^T = A$, $(kA)^T = kA^T$. $(A + B)^T = A^T + B^T$, $(AB)^T = B^T A^T$.
- Co-factor of a_{ij} of $A_{ij} = (-1)^{i+j}\, m_{ij}$ where m_{ij} is the minor of a_{ij}.
- $|AB| = |A| \cdot |B|$ where A and B are square matrices of same order.

TEXTUAL QUESTIONS

Exercise 7.1

1. Construct an $m \times n$ matrix $A = [a_{ij}]$, where a_{ij} is given by

(i) $a_{ij} = \dfrac{(i-2j)^2}{2}$ with $m = 2, n = 3$

(ii) $a_{ij} = \dfrac{|3i-4j|}{4}$ with $m = 3, n = 4$

Solution :

(i) Given $a_{ij} = \dfrac{(i-2j)^2}{2}$ with $m = 2, n = 3$

we need to construct a 2×3 matrix.

$$\therefore a_{11} = \frac{(1-2(1))^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$a_{12} = \frac{(1-2(2))^2}{2} = \frac{(-3)^2}{2} = \frac{9}{2}$$

$$a_{13} = \frac{(1-2(3))^2}{2} = \frac{(-5)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2-2(1))^2}{2} = \frac{0}{2} = 0$$

$$a_{22} = \frac{(2-2(2))^2}{2} = \frac{(-2)^2}{2} = \frac{4}{2}$$

$$a_{23} = \frac{(2-2(3))^2}{2} = \frac{(-4)^2}{2} = \frac{16}{2}$$

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 1/2 & 9/2 & 25/2 \\ 0 & 4/2 & 16/2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 9 & 25 \\ 0 & 4 & 16 \end{pmatrix}$$

(ii) Given $a_{ij} = \dfrac{|3i-4j|}{4}$ with $m = 3, n = 4$.

Let B be a 3×4 matrix with entries as

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$$a_{ij} = \frac{|3i-4j|}{4}$$

$$a_{11} = \frac{|3-4|}{4} = \frac{|-1|}{4} = \frac{1}{4}$$

$$a_{12} = \frac{|3-8|}{4} = \frac{|-5|}{4} = \frac{5}{4}$$

$$a_{13} = \frac{|3-12|}{4} = \frac{|-9|}{4} = \frac{9}{4}$$

$$a_{14} = \frac{|3-16|}{4} = \frac{|-13|}{4} = \frac{13}{4}$$

$$a_{21} = \frac{|3(2)-4(1)|}{4} = \frac{|6-4|}{4} = \frac{2}{4}$$

$$a_{22} = \frac{|3(2)-4(2)|}{4} = \frac{|6-8|}{4} = \frac{2}{4}$$

$$a_{23} = \frac{|3(2)-4(3)|}{4} = \frac{|6-12|}{4} = \frac{6}{4}$$

$$a_{24} = \frac{|3(2)-4(4)|}{4} = \frac{|6-16|}{4} = \frac{10}{4}$$

$$a_{31} = \frac{|3(3)-4(1)|}{4} = \frac{|9-4|}{4} = \frac{5}{4}$$

$$a_{32} = \frac{|3(3)-4(2)|}{4} = \frac{|9-8|}{4} = \frac{1}{4}$$

$$a_{33} = \frac{|3(3)-4(3)|}{4} = \frac{|9-12|}{4} = \frac{3}{4}$$

$$a_{34} = \frac{|3(3)-4(4)|}{4} = \frac{|9-16|}{4} = \frac{7}{4}$$

$$\therefore B = \begin{bmatrix} 1/4 & 5/4 & 9/4 & 13/4 \\ 2/4 & 2/4 & 6/4 & 10/4 \\ 5/4 & 1/4 & 3/4 & 7/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{bmatrix}$$

2. Find the values of p, q, r, and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$$

Solution :

Given $\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & 3/2 & 9 \\ -2 & 8 & -\pi \end{bmatrix}$

Since the matrices are equal, the corresponding entries on both sides are equal.

$\therefore p^2-1=1 \Rightarrow p^2 = 2 \Rightarrow p = \pm\sqrt{2}$ [Equating a_{11}]

$-31-q^3 = -4 \Rightarrow -q^3 = -4+31$ [Equating a_{13}]

$\Rightarrow -q^3 = 27$

$q^3 = -27 = (-3)^3$

$\Rightarrow q = -3$

Also $r+1 = 3/2 \Rightarrow r = 3/2 - 1 = \frac{3-2}{2} = 1/2$ [Equating a_{22}]

$s-1 = -\pi \Rightarrow s = 1-\pi$ [Equating a_{33}]

$p = \pm\sqrt{2},\ q=-3,\ r = 1/2,\ s = 1-\pi.$

3. Determine the value of $x + y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

Solution : Given $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

Equating the corresponding entries on both sides we get,

$2x+y = 7$ [Equating a_{11}] ... (1)

$4x = x+6$ [Equating a_{22}] ... (2)

From (2), $4x - x = 6 \Rightarrow 3x = 6 \Rightarrow x = \frac{6}{3} \Rightarrow x = 2$

Substituting $x = 2$ in (1) we get,

$4+y = 7 \Rightarrow y = 7-4 \Rightarrow y = 3$

$\therefore x+y = 2+3 = 5$

4. Determine the matrices A and B if they satisfy $2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0$ and $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$

Solution : Given $2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0$

$\Rightarrow 2A - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$ ... (1)

Also given $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$ ... (2)

$(1)\times 2 \Rightarrow 4A - 2B = \begin{bmatrix} -12 & 12 & 0 \\ 8 & -4 & -2 \end{bmatrix}$

$(2) \Rightarrow \underset{(-)}{A} \underset{(+)}{-2B} = \underset{(-)}{\begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}}$

Subtracting, $3A = \begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$

$\Rightarrow A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$

Substituting the matrix A in (1) we get,

$\frac{1}{3}\begin{bmatrix} -30 & 20 & -16 \\ 20 & -10 & +10 \end{bmatrix} - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$

$\Rightarrow \frac{1}{3}\begin{bmatrix} -30 & 20 & -16 \\ 20 & -10 & +10 \end{bmatrix} - \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} = B$

$\Rightarrow B = \begin{bmatrix} -10+6 & \frac{20}{3}-6 & \frac{-16}{3}-0 \\ \frac{20}{3}-4 & \frac{-10}{3}+2 & \frac{10}{3}+1 \end{bmatrix}$

$= \begin{bmatrix} -4 & \frac{20-18}{3} & \frac{-16}{3} \\ \frac{20-12}{3} & \frac{-10+6}{3} & \frac{10+3}{3} \end{bmatrix}$

$\Rightarrow B = \begin{bmatrix} -4 & 2/3 & -16/3 \\ 8/3 & -4/3 & 13/3 \end{bmatrix} \therefore B = \frac{1}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then compute A^4.

Solution : Given $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & a+a \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$

$A^4 = A^2.A^2 = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1+0 & 2a+2a \\ 0+0 & 0+1 \end{bmatrix};\ A^4 = \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$

6. Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(i) Show that $A_\alpha A_\beta = A_{(\alpha+\beta)}$

(ii) Find all possible real values of α satisfying the condition $A_\alpha + A_\alpha^T = I$.

Solution : Given $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$A_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

(i) $\therefore A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\left[\begin{matrix} \text{since } \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{matrix}\right]$$

$$A_\alpha . A_\beta = A_{\alpha+\beta}$$

Hence proved.

(ii) Given $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$A_\alpha^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Also, it is given that $A_\alpha + A_\alpha^T = I$

$$\Rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3}.$$

$$\Rightarrow \quad \alpha = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

$[\because \cos\alpha = \cos\theta \Rightarrow \alpha = 2n\pi \pm \theta, n \in \mathbb{Z}]$

$$\therefore \alpha = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$$

7. If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that $(A-2I)(A-3I) = 0$, find the value of x.

Solution : Given $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$

Also, $(A-2I)(A-3I) = 0$

$$\therefore A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 2-0 \\ -1-0 & x-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4-3 & 2-0 \\ -1-0 & x-9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & x-9 \end{bmatrix}$$

$$\therefore (A-2I)(A-3I) = \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & x-9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-2 & 4+2(x-9) \\ -1-1(x-2) & -2+(x-2)(x-9) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 4+2x-18 \\ -1-x+2 & -2+x^2-11x+18 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2x-14 \\ -x+1 & x^2-11x+16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding entries we get,

$\Rightarrow \quad 2x - 14 = 0 \quad$ or $\quad -x + 1 = 0$

$\Rightarrow \quad 2x = 14 \quad$ or $\quad -x = -1$

$\Rightarrow \quad x = 7 \quad$ or $\quad x = 1$

Since $x = 1$ alone satisfies the equation $(A-2I)(A-3I) = 0$, we get $x = 1$.

8. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that A^2 is a unit matrix.

Solution : Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A^2$ is a unit matrix.

9. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI = 0$, find the value of k. **[Hy - 2018]**

Solution : Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Also, $A^3 - 6A^2 + 7A + kI = 0$ ... (1)

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \quad ... (2)$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} \quad ... (3)$$

Substituting (2) and (3) in (1) we get,

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 21-30+7+k & 0 & 34-48+14+0 \\ \cancel{12}-\cancel{12}+0+0 & 8-24+14+k & 23-30+7+0 \\ 34-48+14+0 & 0+0+0+0 & 55-78+21+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2+k & 0 & 0 \\ 0 & -2+k & 0 \\ 0 & 0 & -2+k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Equating the corresponding entries both sides, we get

$$-2+k = 0$$
$$k = 2$$

10. Give your own examples of matrices satisfying the following conditions in each case:

(i) A and B such that AB ≠ BA.

(ii) A and B such that AB = 0 = BA, A ≠ 0 and B ≠ 0.

(iii) A and B such that AB = 0 and BA ≠ 0.

Solution : (i) Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2+4 & -\cancel{2}+\cancel{2}+12 \\ 12+2+5 & -4+2+15 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 19 & 13 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 3-1 & 12-5 \\ 4+8 & 2+2 & 8+10 \\ 2+12 & 1+3 & 4+15 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 7 \\ 12 & 4 & 18 \\ 14 & 4 & 19 \end{bmatrix}$$

$\therefore AB \neq BA$

(ii) Let $A = \begin{bmatrix} 0 & -3 \\ 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & -3 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0+0 & -12+12 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $AB = 0 = BA$ and $A \neq 0$, $B \neq 0$.

(iii) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore AB = 0$ and $BA \neq 0$

11. Show that $f(x)f(y) = f(x+y)$, where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Solution : Given $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$f(x).f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x\cos y & -\cos x\sin y - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x+y)$$

[$\because \cos(x+y) = \cos x \cos y - \sin x \sin y = f(x+y)$
$\sin(x+y) = \sin x \cos y + \cos x \sin y$]

12. If A is a square matrix such that $A^2 = A$, find the value of $7A - (I + A)^3$.

Solution : Given A is a square matrix and $A^2 = A$

Consider $7A - (I+A)^3 = 7A - (I^3 + 3I^2A + 3IA^2 + A^3)$

[$\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

$= 7A - (I + 3A + 3A^2 + A^2.A)$

[$\because I^3 = I, I^2 = I$]

$= 7A - (I + 3A + 3A + A.A)$

[$\because A^2 = A$]

$= 7A - (I + 7A)$

$= \cancel{7A} - I - \cancel{7A} = -I$

13. Verify the property $A(B+C) = AB + AC$, when the matrices A, B, and C are given by

$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$.

Solution : Given $A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$

and $C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$

$$B + C = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 1+7 \\ -1+2 & 0+1 \\ 4+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$\therefore A(B+C) = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+0-15 & 16+0-3 \\ 7+4+25 & 8+4+5 \end{bmatrix}$$

$$\text{LHS} = A(B+C) = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \quad \dots (1)$$

$$AB = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0-12 & 2+0-6 \\ 3-4+20 & 1+0+10 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8+0-3 & 14+0+3 \\ 4+8+5 & 7+4-5 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$RHS = AB + AC$$

$$= \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6+5 & -4+17 \\ 19+17 & 11+6 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), A(B + C) = AB + AC.

14. Find the matrix A which satisfies the matrix relation A $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

Solution : Given A$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

The order of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is 2 × 3 and the order of the matrix $\begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$ is also 2 × 3.

∴ A must be of order 2 × 2.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$a + 4b = -7 \quad \ldots (1)$$
$$2a + 5b = -8 \quad \ldots (2)$$
$$c + 4d = 2 \quad \ldots (3)$$
$$2c + 5d = 4 \quad \ldots (4)$$

(1) × 2 ⇒ $2a + 8b = -14$

(2) ⇒ $2a + 5b = -8$

(–) (–) (+)

$3b = -6 \Rightarrow b = -2$

Substituting $b = -2$ in (1) we get,

$a - 8 = -7 \Rightarrow a = -7 + 8 \Rightarrow a = 1$

(3) × 2 ⇒ $2c + 8d = 4$

(4) ⇒ $2c + 5d = 4$

(–) (–) (–)

$3d = 0 \Rightarrow d = 0$

Substituting $d = 0$ in (3) we get,

$c = 2$

$$\therefore A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$$

15. If $A^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$, verify the following

(i) $(A + B)^T = A^T + B^T = B^T + A^T$

(ii) $(A - B)^T = A^T - B^T$ (iii) $(B^T)^T = B$.

Solution : Given $A^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$

(i) Verify $(A + B)^T = A^T + B^T = B^T + A^T$

$$(A^T)^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}^T \Rightarrow A = \begin{bmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$\text{Now, } A + B = \begin{bmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & 3 \\ 12 & 5 & 1 \end{bmatrix}$$

$$\therefore (A + B)^T = \begin{bmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{bmatrix} \quad \ldots (1)$$

$$B^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\therefore A^T + B^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{bmatrix} \quad \ldots (2)$$

$$B^T + A^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{bmatrix} \quad \ldots (3)$$

From (1), (2) and (3), $(A + B)^T = A^T + B^T = B^T + A^T$

(ii) Verify $(A-B)^T = A^T - B^T$

$$A-B = \begin{bmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -2 & -5 & 5 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix} \quad \ldots (4)$$

$$A^T - B^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix} \quad \ldots (5)$$

From (4) and (5), $(A-B)^T = A^T - B^T$

(iii) Verify $(B^T)^T = B$

$$\text{Given } B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix} \quad \ldots (6)$$

$$\therefore B^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\text{Also, } (B^T)^T = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix} \quad \ldots (7)$$

From (6) and (7), $(B^T)^T = B$

16. If A is a 3 × 4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B?

Solution : Given A is a 3 × 4 matrix.

A^T is a 4 × 3 matrix.

A^TB and BA^T are defined.

To define A^TB, B must be a 3 × 4 matrix.

Also to define BA^T, B must be a 3 × 4 matrix.

Hence, the order of matrix B is (3 × 4)

17. Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ **and (ii)** $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$.

Solution : (i) Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$$\text{Let } P = \frac{1}{2}\left(A + A^T\right)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix}$$

$$\Rightarrow \quad P^T = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} = P$$

$\therefore$ P is a symmetric matrix.

$$\text{Let } Q = \frac{1}{2}\left[A - A^T\right]$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$Q^T = \frac{1}{2}\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -Q$$

$\therefore$ Q is a skew-symmetric matrix.

$$\text{Now } A = P + Q = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

Thus A is expressed as the sum of symmetric and skew-symmetric matrices.

$$\text{(ii) Let } B = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}\left[B + B^T\right] = \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$\Rightarrow \quad P^T = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = P$$

$\therefore$ P is a symmetric matrix.

$$\text{Let } Q = \frac{1}{2}\left[B - B^T\right] = \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

$$\Rightarrow \quad Q^T = \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix} = -Q$$

$\therefore$ Q is a skew-symmetric matrix.

$$\text{Now B} = P + Q$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

Thus, B is expressed as the sum of a symmetric and a skew-symmetric matrix.

18. Find the matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$.

Solution : Given $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times2}$ $A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}_{3\times3}$

A^T is a matrix of order 2 × 3.

Let $A^T = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a & b & c \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$a = 1, b = 2, c = -5$ and $2a - d = -1$

$\Rightarrow 2 - d = -1 \Rightarrow 2 + 1 = d \Rightarrow d = 3$

$2b - e = -8 \Rightarrow 4 - e = -8 \Rightarrow 4 + 8 = e$

$\Rightarrow e = 12$

$2c - f = -10 \Rightarrow -10 - f = -10 \Rightarrow f = 0$

$$\therefore A^T = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 12 & 0 \end{bmatrix}$$

$$\Rightarrow (A^T)^T = A = \begin{bmatrix} 1 & 3 \\ 2 & 12 \\ -5 & 0 \end{bmatrix}$$

19. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ is a matrix such that $AA^T = 9I$, find the values of x and y.

Solution : Given $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix}$

$$\text{Also, } AA^T = 9I$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+4+4 & 2+2-4 & x+4+2y \\ 2+2-4 & 4+1+4 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & x+2y+4 \\ 0 & 9 & 2x-2y+2 \\ x+2y+4 & 2x-2y+2 & x^2+y^2+4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$x + 2y + 4 = 0$

$2x - 2y + 2 = 0$

$\Rightarrow x + 2y = -4$ ...(1)

$\Rightarrow 2x - 2y = -2$ .. (2)

$3x = -6 \Rightarrow x = -2$

Substituting $x = -2$ in (1) we get,

$-2 + 2y = -4$

$\Rightarrow 2y = -4 + 2 = -2$

$\Rightarrow y = -1$

Hence, $x = -2, y = -1$

20. (i) For what value of x, the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is skew-symmetric. **[Hy - 2018]**

(ii) If $\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of p, q, and r.

Solution :

(i) Given $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

$$\Rightarrow \quad A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & x^3 & 0 \end{bmatrix}$$

Since A is a skew-symmetric matrix.

$$A^T = -A$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & x^3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & x^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -x^3 \\ -2 & 3 & 0 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$x^3 = 3 \Rightarrow x = \sqrt[3]{3}$$

(ii) Let $B = \begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$

$$\Rightarrow \quad B^T = \begin{bmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

Since B is a skew-symmetric matrix,

$$B^T = -B$$

$$\begin{bmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -p & -3 \\ -2 & -q^2 & 1 \\ -r & -1 & 0 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$2 = -p \Rightarrow p = -2$$
$$r = -3$$
$$q^2 = -q^2 \Rightarrow 2q^2 = 0$$
$$\Rightarrow \quad q^2 = \frac{0}{2} = 0 \Rightarrow q = 0$$

Hence, $p = -2$, $q = 0$ and $r = -3$

21. Construct the matrix $A = [a_{ij}]_{3\times 3}$, where $a_{ij} = i - j$. State whether A is symmetric or skew-symmetric.

Solution : Given $a_{ij} = i - j$

Let $A = [a_{ij}]_{3\times3}$

$\therefore a_{11} = 1-1 = 0 \quad a_{21} = 2-1 = 1 \quad a_{31} = 3-1 = 2$

$a_{12} = 1-2 = -1 \quad a_{22} = 2-2 = 0 \quad a_{32} = 3-2 = 1$

$a_{13} = 1-3 = -2 \quad a_{23} = 2-3 = -1 \quad a_{33} = 3-3 = 0$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \therefore A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = -A$$

Since $A^T = -A$, A is a skew-symmetric matrix.

22. Let A and B be two symmetric matrices. Prove that AB = BA if and only if AB is a symmetric matrix.

Solution : A and B are symmetric

If $(AB)^T = AB$

$\Rightarrow \quad B^T A^T = AB \quad [\because (AB)^T = B^T A^T]$

$\Rightarrow \quad BA = AB$

$[\because$ A and B are symmetric matrices $\Rightarrow B^T = B$ and $A^T = A]$

Hence proved.

23. If A and B are symmetric matrices of same order, prove that

(i) AB + BA is a symmetric matrix.

(ii) AB – BA is a skew-symmetric matrix.

Solution : Given A and B are symmetric matrices

(i) $\Rightarrow \quad A^T = A$ and $B^T = B$ ... (1)

To prove that (AB + BA) is a symmetric matrix.

Consider $(AB + BA)^T = (AB)^T + (BA)^T$

$= B^TA^T + A^TB^T$

$[\because (AB)^T = B^TA^T]$

$= BA + AB$ [using (1)]

$= AB + BA$

$\Rightarrow \quad (AB+BA)^T = AB+BA$

$\therefore$ (AB + BA) is a symmetric matrix.

(ii) Given A and B are symmetric matrices

$\Rightarrow \quad A^T = A$ and $B^T = B$ ... (2)

To prove that (AB – BA) is a skew-symmetric matrix.

Consider $(AB-BA)^T = (AB)^T - (BA)^T$

$= B^TA^T - A^TB^T$ $[\because (AB)^T = B^TA^T]$

$= BA - AB$ [using (2)]

$= -(AB-BA)$

$\Rightarrow \quad (AB-BA)^T = -(AB-BA)$

$\therefore$ (AB – BA) is a skew-symmetric matrix.

24. A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins and almonds.
Pack I contains 100 gm of cashew nuts, 100 gm of raisins and 50 gm of almonds.
Pack-II contains 200 gm of cashew nuts, 100 gm of raisins and 100 gm of almonds.
Pack-III contains 250 gm of cashew nuts, 250 gm of raisins and 150 gm of almonds.
The cost of 50 gm of cashew nuts is ₹ 50/-, 50 gm of raisins is ₹10/-, and 50 gm of almonds is ₹ 60/-. What is the cost of each gift pack?

Solution : Gift pack matrix is as follows:

	I	II	III
Weight of Cashew nuts	100	200	250
Weight of Raisins	100	100	250
Weight of Almonds	50	100	150

Let us consider 50 gm of cashew nuts as one packet, 50 gm of raisins as one packet and 50 gm of almonds as one packet, we get the matrix as

	I	II	III
No. of packets of cashewnuts	2	4	5
No. of packets of raisins	2	2	5
No. of packets of almonds	1	2	3

$= A$

Given cost matrix is $\begin{bmatrix} 50 & 10 & 60 \end{bmatrix} = B$

$\therefore$ Cost of gift pack

$$= AB = \begin{bmatrix} 50 & 10 & 60 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 100+20+60 \\ 200+20+120 \\ 250+50+180 \end{bmatrix} = \begin{bmatrix} 180 \\ 340 \\ 480 \end{bmatrix}$$

$\therefore$ Cost of I gift pack = ₹ 180

Cost of II gift pack = ₹ 340 and cost of III gift pack = ₹ 480

EXERCISE 7.2

1. Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

Solution : Let $A = \begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix}$

Applying $C_2 \to C_2 + C_3$ we get,

$$A = \begin{vmatrix} s & a^2+b^2+c^2 & b^2+c^2 \\ s & a^2+b^2+c^2 & c^2+a^2 \\ s & a^2+b^2+c^2 & a^2+b^2 \end{vmatrix}$$

Taking 's' common from C_1 and $(a^2+b^2+c^2)$ common from C_2 we get

$$A = s(a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & b^2+c^2 \\ 1 & 1 & c^2+a^2 \\ 1 & 1 & a^2+b^2 \end{vmatrix}$$

$$= s(a^2+b^2+c^2)\,(0) = 0 \quad [\because C_1 \equiv C_2]$$

Hence, $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

2. Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0.$

Solution : Applying $R_1 \to aR_1$, $R_2 \to bR_2$ and $R_3 \to cR_3$ we get,

$$A = \begin{vmatrix} ab+ac & abc & ab^2c^2 \\ bc+ab & abc & a^2bc^2 \\ ac+bc & abc & a^2b^2c \end{vmatrix}$$

Taking out (abc) common from C_2 and C_3 we get,

$$= (abc)^2 \begin{vmatrix} ab+ac & 1 & bc \\ bc+ab & 1 & ac \\ ac+bc & 1 & ab \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$ we get,

$$= (abc)^2 \begin{vmatrix} ab+bc+ca & 1 & bc \\ ab+bc+ca & 1 & ac \\ ab+bc+ca & 1 & ab \end{vmatrix}$$

Taking out $(ab + bc + ca)$ common from C_1, we get

$$A = a^2b^2c^2\,(ab + bc + ca) \begin{vmatrix} 1 & 1 & bc \\ 1 & 1 & ac \\ 1 & 1 & ab \end{vmatrix}$$

$$= a^2b^2c^2\,(ab + bc + ca)\,(0) = 0 \qquad [\because C_1 \equiv C_2]$$

3. **Prove that** $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2\,b^2\,c^2.$

Solution : LHS $= \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$

Taking out a, b, c common from C_1, C_2 and C_3 respectively we get,

$$\text{LHS} = (abc) \begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$$= (abc) \begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix}$$

$$= 2abc \begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ b+c & b+c & c \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_1$ we get,

$$= 2abc \begin{vmatrix} a+c & -a & 0 \\ a+b & -a & -b \\ b+c & 0 & -b \end{vmatrix}$$

Applying $C_1 \to C_2 + C_1 + C_3$ we get,

$$\text{LHS} = 2abc \begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

Taking c, a, b common from C_1, C_2 and C_3 respectively.

$$= 2a^2b^2c^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

Expanding along R_1 we get,

$$= 2a^2b^2c^2 \left[1 \begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} \right]$$

$$= 2a^2b^2c^2\,[(1 - 0) + (0 + 1)]$$

$$= 2a^2b^2c^2\,[2]$$

$$= 4a^2b^2c^2 = \text{RHS}$$

Hence proved.

4. **Prove that** $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$

Solution : LHS $= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Taking out a, b, c common from R_1, R_2 and R_3 respectively.

$$\text{LHS} = abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ we get,

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1}{c}+1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$ we get,

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & \frac{1}{c}+1 \end{vmatrix}$$

Expanding along R_1 we get,

$$= abc\,(1 + 1/a + 1/b + 1/c)\left[0+0+1\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}\right]$$

$$= abc\,(1 + 1/a + 1/b + 1/c)\,[1]$$

$$= abc\,(1 + 1/a + 1/b + 1/c) = \text{RHS}$$

Hence Proved.

5. Prove that $\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0.$

Solution : $\text{LHS} = \begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix}$

Applying $C_2 \to C_2 + C_3$ we get,

$$= \begin{vmatrix} \sec^2\theta & 1+\tan^2\theta & 1 \\ \tan^2\theta & -1+\sec^2\theta & -1 \\ 38 & 38 & 2 \end{vmatrix} = \begin{vmatrix} \sec^2\theta & \sec^2\theta & 1 \\ \tan^2\theta & \tan^2\theta & -1 \\ 38 & 38 & 2 \end{vmatrix}$$

$[\because\ 1 + \tan^2\theta = \sec^2\theta \text{ and } \sec^2\theta - 1 = \tan^2\theta]$

$= 0 \qquad [\because C_1 \equiv C_2] = \text{RHS}$

Hence Proved.

6. Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0.$

Solution : $\text{LHS} = \begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = \begin{vmatrix} x & y & z \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ a & b & c \end{vmatrix}$ [By Property 7]

$$= 0 + 2\begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix} \qquad [\because R_1 \equiv R_2]$$

$$= 0 + 2(0) = 0 = \text{RHS} \qquad [\because R_1 \equiv R_3]$$

7. Write the general form of a 3 × 3 skew-symmetric matrix and prove that its determinant is 0.

Solution : A square matrix $A = [a_{ij}]_{3\times3}$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j and the elements on the main diagonal of a skew-symmetric matrix are zero.

$$\therefore A = \begin{vmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{vmatrix}$$

Expanding along R_1 we get,

$$|A| = 0 - a_{12}\begin{vmatrix} -a_{12} & a_{23} \\ -a_{13} & 0 \end{vmatrix} + a_{13}\begin{vmatrix} -a_{12} & 0 \\ -a_{13} & -a_{23} \end{vmatrix}$$

$$= -a_{12}\,(a_{13}\,a_{23}) + a_{13}(a_{12}a_{23})$$

$$= -a_{12}\,\cancel{a_{13}}\,a_{23} + a_{12}\,\cancel{a_{13}}\,a_{23} = 0$$

Hence the determinant of a skew-symmetric matrix is 0.

8. If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$**, prove that** a, b, c **are in G.P. or** α **is a root of** $ax^2 + 2bx + c = 0$**.**

Solution : Given $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$

Expanding along R_3 we get,

$$(a\alpha + b)\begin{vmatrix} b & a\alpha+b \\ c & b\alpha+c \end{vmatrix} - (b\alpha + c)\begin{vmatrix} a & a\alpha+b \\ b & b\alpha+c \end{vmatrix} + 0 = 0$$

$\Rightarrow -(a\alpha + b)\,(b^2\alpha + \cancel{bc} - ac\alpha - \cancel{bc}) - (b\alpha + c)\,(\cancel{ab}\alpha + ac - \cancel{ab}\alpha - b^2) = 0$

$\Rightarrow (a\alpha + b)\,(b^2\alpha - ac\alpha) - (b\alpha + c)\,(ac - b^2) = 0$

$\Rightarrow \alpha(a\alpha + b)\,(b^2 - ac) + (b\alpha + c)\,(b^2 - ac) = 0$

$\Rightarrow (b^2 - ac)\,(a\alpha^2 + b\alpha + b\alpha + c) = 0$

$\Rightarrow (b^2 - ac)(a\alpha^2 + 2b\alpha + c) = 0$

$\Rightarrow b^2 - ac = 0$ or $a\alpha^2 + 2b\alpha + c = 0$

$\Rightarrow ac = b^2$ or $a\alpha^2 + 2b\alpha + c = 0$

$\Rightarrow a, b, c$ are in G.P. (or) α is a root of $ax^2 + 2bx + c = 0$

9. **Prove that** $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$ **[Hy - 2018]**

Solution : LHS $= \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$- \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ [By property 7]

Multiplying and dividing R_1, R_2 and R_3 of second determinant by a, b, c respectively.

$$\text{LHS} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

Taking abc common from C_3 of second determinant

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Applying $C_2 \leftrightarrow C_3$ in the second determinant,

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_2$ in the second determinant

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 = \text{RHS}$$

Hence proved.

10. **If a, b, c are p^{th}, q^{th} and r^{th} terms of an A.P, find the value of** $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$.

Solution : Given $a = t_p$, $b = t_q$ and $c = t_r$.

Let A be the first term and l be the last term of the A.P.

$$\therefore a = \frac{p}{2}(A+l),\ b = \frac{q}{2}(A+l),\ c = \frac{r}{2}(A+l) \qquad \dots (1)$$

Consider $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \frac{p}{2}(A+l) & \frac{q}{2}(A+l) & \frac{r}{2}(A+l) \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

using (1)

Taking $\left(\frac{A+l}{2}\right)$ common from R_1 we get,

$$= \frac{A+l}{2}\begin{vmatrix} p & q & r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = \frac{A+l}{2}(0) = 0 \qquad [\because R_1 \equiv R_2]$$

$$\therefore \begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

11. **Show that** $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ **is divisible by x^4.**

Solution : LHS $= \begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$

Multiplying R_1,R_2, R_3 by a, b, c respectively, and dividing the determinant by abc we get,

$$= \frac{1}{abc}\begin{vmatrix} a^3+ax^2 & a^2b & a^2c \\ ab^2 & b^3+bx^2 & b^2c \\ ac^2 & bc^2 & c^3+cx^2 \end{vmatrix}$$

Taking a, b, c common from C_1, C_2 and C_3 we get,

$$= \frac{abc}{abc}\begin{vmatrix} a^2+x^2 & a^2 & a^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$ we get,

$$= \begin{vmatrix} a^2+b^2+c^2+x^2 & a^2+b^2+c^2+x^2 & a^2+b^2+c^2+x^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

Taking $(a^2 + b^2 + c^2 + x^2)$ common from R_1, we get

$$\text{LHS} = (a^2+b^2+c^2+x^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$ we get,

$$= (a^2+b^2+c^2+x^2)\begin{vmatrix} 0 & 0 & 1 \\ -x^2 & x^2 & b^2 \\ 0 & -x^2 & c^2+x^2 \end{vmatrix}$$

Taking out x^2 common from C_1 and C_2 we get,

$$= x^4(a^2+b^2+c^2+x^2)\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b^2 \\ 0 & -1 & c^2+x^2 \end{vmatrix}$$

Expanding along R_1 we get,

$$\text{LHS} = x^4(a^2+b^2+c^2+x^2)\left[1\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}\right]$$

$$= x^4(a^2+b^2+c^2+x^2)\,(1)$$

$= x^4(a^2 + b^2 + c^2 + x^2)$ which is divisible by x^4.

12. If a, b, c are all positive, and are p^{th}, q^{th} and r^{th} terms of a G.P., show that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$

Solution : Given a, b, c are p^{th}, q^{th} and r^{th} terms of a G.P.

Let A be the first term and R be the common ratio of the G.P.

$$\therefore a = \text{AR}^{p-1} \Rightarrow \log a = \log \text{A} + (p-1)\log \text{R}$$

$$b = \text{AR}^{q-1} \Rightarrow \log b = \log \text{A} + (q-1)\log \text{R}$$

$$c = \text{AR}^{r-1} \Rightarrow \log c = \log \text{A} + (r-1)\log \text{R}$$

$$\therefore \text{Let A} = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$$

$$\therefore \text{A} = \begin{vmatrix} \log \text{A} + (p-1)\log \text{R} & p & 1 \\ \log \text{A} + (q-1)\log \text{R} & q & 1 \\ \log \text{A} + (r-1)\log \text{R} & r & 1 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_3$ we get,

$$\text{A} = \begin{vmatrix} \log \text{A} + (p-1)\log \text{R} & p-1 & 1 \\ \log \text{A} + (q-1)\log \text{R} & q-1 & 1 \\ \log \text{A} + (r-1)\log \text{R} & r-1 & 1 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - (\log \text{A})\,C_3 - (\log \text{R})C_2$ we get,

$$\text{A} = \begin{vmatrix} 0 & p-1 & 1 \\ 0 & q-1 & 1 \\ 0 & r-1 & 1 \end{vmatrix} = 0$$

$\therefore \text{A} = 0$ Hence proved.

13. Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ if $x, y, z \neq 1$.

Solution : Let $\text{A} = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

Given $x, y, z \neq 1$

Expanding along R_1 we get,

$$\text{A} = 1\begin{vmatrix} 1 & \log_y z \\ \log_z y & 1 \end{vmatrix} - \log_x y\begin{vmatrix} \log_y x & \log_y z \\ \log_z x & 1 \end{vmatrix} + \log_x z\begin{vmatrix} \log_y x & 1 \\ \log_z x & \log_z y \end{vmatrix}$$

$= 1 - \log_y z \,.\, \log_z y - \log_x y \;= 1 - \log_y z \,.\, \log_z y$

$(\log_y x - \log_y z.\, \log_z x) + \log_x z\,(\log_y x \log_z y - \log_z x)$

$= 1 - 1 - \log_x y\,(\log_y x - \log_y x) + \log_x z\,(\log_z x - \log_z x)$

$[\because \log_z x.\log_x z = 1 \text{ and } \log_y z.\log_z x = \log_y x]$

$= 0 - \log_x y\,(0) + \log_x z\,(0) = 0$

$$\therefore \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = 0$$

14. If $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$, prove that $\sum_{k=1}^{n} \det(A^k) = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$.

Solution : Given $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$

$$\Rightarrow \quad |A| = \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^2$$

$$\text{Also } A^2 = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\right)^2 & \alpha \\ 0 & \left(\frac{1}{2}\right)^2 \end{bmatrix}$$

$$\therefore |A^2| = \left(\frac{1}{2}\right)^4$$

$$\therefore \sum_{k=1}^{n} \det\left(A^k\right) = \det(A) + \det(A^2) + \det(A^3) + \dots\dots + \det(A^n)$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \dots\dots\dots + \left(\frac{1}{2}\right)^{2n}$$

$$= \left(\frac{1}{2}\right)^2\left[1 + \left(\frac{1}{2}\right)^2 + \dots\dots\dots + \left(\frac{1}{2}\right)^{2(n-1)}\right] \quad \dots (1)$$

$1 + \left(\frac{1}{2}\right)^2 + \dots\dots\dots\dots + \left(\frac{1}{2}\right)^{2(n-1)}$ is a G.P. with $a = 1$ and $r = \left(\frac{1}{2}\right)^2$.

$$\therefore S_n = \left(\frac{1}{2}\right)^2 \left(\frac{1 - \left(\frac{1}{2^2}\right)^n}{1 - \left(\frac{1}{2}\right)^2}\right)$$

$$= \frac{1 - \frac{1}{4^n}}{1 - \frac{1}{4}} \times \left(\frac{1}{2}\right)^2$$

$$= \frac{1 - \frac{1}{4^n}}{\frac{3}{4}} \left(\frac{1}{2}\right)^2$$

$$\left[\because S_n = \frac{a\left(1 - r^n\right)}{1 - r}\right]$$

$$\therefore \sum_{k=1}^{n} \det\left(A^k\right) = \left(\frac{1}{2}\right)^2 \left[\frac{1 - \frac{1}{4^n}}{\frac{3}{4}}\right] \quad \text{[From (1)]}$$

$$= \frac{1}{4} \times \frac{4}{3}\left[1 - \frac{1}{4^n}\right] = \frac{1}{3}\left[1 - \frac{1}{4^n}\right]$$

Hence proved.

15. Without expanding, evaluate the following determinants :

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

(ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Solution : (i) Let $A = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

Taking $(3x)$ common from R_3 we get,

$$A = 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 3x\,(0) = 0$$

$[\because R_1 \equiv R_3]$

(ii) Let $B = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$B = \begin{vmatrix} x-z & y-x & z+x \\ z-x & x-y & y \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -(z-x) & -(x-y) & z+x \\ z-x & x-y & y \\ 0 & 0 & 1 \end{vmatrix}$$

Taking $(z-x)$ and $(x-y)$ common from C_1 and C_2 we get,

$$= (z-x)(x-y)\begin{vmatrix} -1 & -1 & 1 \\ 1 & 1 & y \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3 we get,

$$B = (z-x)(x-y)\left[1\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}\right]$$

$$= (z-x)(x-y)(-1+1) = 0$$

$$\therefore B = 0$$

16. If A is a square matrix and | A | = 2, find the value of $|AA^T|$.

Solution : Given A is a square matrix and

$$|A| = 2$$

$$\therefore |AA^T| = |A|\,|A^T| = |A|\,.\,|A| \quad [\because |A|^T = |A|]$$

$$= 2(2) = 4 \qquad \text{By property 1}$$

17. If A and B are square matrices of order 3 such that | A | = –1 and |B| = 3, find the value of |3AB|.

Solution : Given A and B are square matrices of order 3.

$$\text{Also, } |A| = -1 \text{ and } |B| = 3$$

$$\text{Consider } |3AB| = 3^3|A|.|B|$$

$$= 27(-1)(3) = -81$$

[$\because$ A is a square matrix of order 3]

$$\therefore |3AB| = -81$$

18. If $\lambda = -2$, determine the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$.

Solution : Given $\lambda = -2$

$$\text{Let } A = \begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4 & 1 \\ 4 & 0 & 13 \\ -1 & -13 & 0 \end{vmatrix}$$

[Put $\lambda = -2$]

Since $a_{12} = -a_{21}$, $a_{13} = -a_{31}$, $a_{23} = -a_{32}$ and the elements in the main diagonal are zero, A is a skew-symmetric matrix.

We know that, determinant of a skew-symmetric matrix is zero.

$$\therefore |A| = 0$$

19. Determine the roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$.

Solution :

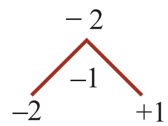
$$\text{Let } A = \begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix}$$

$$\text{Given } A = 0$$

$$\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ we get,

$$\begin{vmatrix} 0 & 6 & 15 \\ 0 & -2-2x & 5-5x^2 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$



Expanding along C_1 we get,

$$0 + 0 + 1\begin{vmatrix} 6 & 15 \\ -2-2x & 5-5x^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 6 & 15 \\ -2-2x^2 & 5-5x^2 \end{vmatrix} = 0$$

$$\Rightarrow 6(5-5x^2) - 15(-2-2x) = 0$$

$$\Rightarrow 30 - 30x^2 + 30 + 30x = 0$$

$$\Rightarrow -30x^2 + 30x + 60 = 0$$

Dividing by – 30 we get,

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \text{ or } -1$$

Hence the roots are –1, 2.

20. Verify that det(AB) = (det A) (det B) for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$.

Solution :

$$\text{Given } A = \begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{vmatrix}$$

$$\text{and } B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6-18 & 12+12-14 & 12+0-10 \\ 1+0+63 & 3+0+49 & 3+0+35 \\ 2-6-45 & 6+12-35 & 6+0-25 \end{bmatrix}$$

$$\det(AB) = \begin{vmatrix} -20 & 10 & 2 \\ 64 & 52 & 38 \\ -49 & -17 & -19 \end{vmatrix}$$

Expanding along R_1 we get,

$$\det(AB) = -20\begin{vmatrix} 52 & 38 \\ -17 & -19 \end{vmatrix} - 10 \times \begin{vmatrix} 64 & 38 \\ -49 & -19 \end{vmatrix} + 2\begin{vmatrix} 64 & 52 \\ -49 & -17 \end{vmatrix}$$

$= -20(-988+646) - 10(-1216+1862) + 2(-1088+2548)$

$= -20(-342) - 10(646) + 2(1460)$

$= -6840 - 6460 - 292$

$= 3300$ ... (1)

$$|A| = \begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= 4\begin{vmatrix} 0 & 7 \\ 3 & -5 \end{vmatrix} - 3\begin{vmatrix} 1 & 7 \\ 2 & -5 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$$

$= 4(0-21) - 3(-5-14) - 2(3+0) = -84 + 57 - 6 = -33$

$$|B| = \begin{vmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 0 \\ 7 & 5 \end{vmatrix} - 3\begin{vmatrix} -2 & 0 \\ 9 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 4 \\ 9 & 7 \end{vmatrix}$$

$= 1(20+0) - 3(-10+0) + 3(-14-36) = 20 + 30 - 150$

$= -100$

$\therefore |A|\,|B| = -33(-100) = 3300$ ... (2)

From (1) and (2) det (AB) = det (A) det (B)

21. Using cofactors of elements of second row, evaluate | A |, where A = $\begin{vmatrix} \mathbf{5} & \mathbf{3} & \mathbf{8} \\ \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{vmatrix}$.

Solution : Given $A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Co-factor of 2 $= A_{21} = (-1)^{1+2}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix}$

$= -(9-16) = 7$

Co-factor of 0 $= A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix}$

$= 15 - 8 = 7$

Co-factor of 1 $= A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$

$= -(10-3) = -7$

$\therefore |A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$

$= 2(7) + 0(7) + 1(-7)$

$= 14 - 7 = 7$

Exercise 7.3

Solve the following problems by using Factor Theorem :

1. Show that $\begin{vmatrix} \boldsymbol{x} & \boldsymbol{a} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{x} & \boldsymbol{a} \\ \boldsymbol{a} & \boldsymbol{a} & \boldsymbol{x} \end{vmatrix} = (\boldsymbol{x}-\boldsymbol{a})^2(\boldsymbol{x}+2\boldsymbol{a})$**.**

Solution : Let $A = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$ ... (1)

Putting $x = a$ in (1) we get,

$$A = \begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix} = 0$$

$[\because R_1 \equiv R_2 \equiv R_3]$

$\therefore (x-a)^2$ is a factor of A.

Putting $x = -2a$ in (1) we get,

$$A = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$$A = \begin{vmatrix} 0 & a & a \\ 0 & -2a & a \\ 0 & a & -2a \end{vmatrix} = 0$$

$\therefore (x + 2a)$ is also a factor of A.

Since the leading diagonal of A is of degree 3, only 3 factors are available and their may be a constant k.

$$\therefore A = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = k(x-a)^2 (x+2a)$$

Putting $x = -a$ in the above equation, we get

$$\begin{vmatrix} -a & a & a \\ a & -a & a \\ a & a & -a \end{vmatrix} = k(-a-a)^2(-a+2a)$$

$$\Rightarrow \begin{vmatrix} 0 & a & a \\ 0 & -a & a \\ 2a & a & -a \end{vmatrix} = k(4a^2)(a)$$

[Applying $C_1 \to C_1 + C_2$]

$$\Rightarrow 2a \begin{vmatrix} a & a \\ -a & a \end{vmatrix} = 4ka^3 \text{ [Expanding along } C_1]$$

$$\Rightarrow 2a(a^2 + a^2) = 4ka^3$$

$$\Rightarrow 2a(2a^2) = 4ka^3$$

$$\Rightarrow \cancel{4}\cancel{a}^3 = \cancel{4}k\cancel{a}^3 \quad \Rightarrow k = 1$$

$$\therefore A = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$$

2. Show that $\begin{vmatrix} \mathbf{b+c} & \mathbf{a-c} & \mathbf{a-b} \\ \mathbf{b-c} & \mathbf{c+a} & \mathbf{b-a} \\ \mathbf{c-b} & \mathbf{c-a} & \mathbf{a+b} \end{vmatrix} = \mathbf{8\,abc}$.

Solution :

$$\text{Let } A = \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}$$

Putting $a = 0$ in (1) we get,

$$A = \begin{vmatrix} b+c & -c & -b \\ b-c & c & b \\ c-b & c & b \end{vmatrix} = 0$$

[$\because C_2 \propto C_3$]

$(a - 0) = a$ is a factor of A.

Putting $b = 0$ in (1) we get,

$$A = \begin{vmatrix} c & a-c & a \\ -c & c+a & -a \\ c & c-a & a \end{vmatrix} = 0$$

[$\because C_1 \propto C_3$]

$(b - 0) = b$ is a factor of A.

Putting $c = 0$ in (1) we get,

$$A = \begin{vmatrix} b & a & a-b \\ b & a & b-a \\ -b & -a & a+b \end{vmatrix} = 0$$

[$\because C_1 \propto C_2$]

$\therefore (c - 0) = c$ is a factor of A.

Since the leading diagonal A is of degree 3, only 3 factors are available and there may exist a constant k.

$$\therefore \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = k(abc)$$

Putting $a = 1$, $b = 1$ and $c = 1$ in the above equation, we get

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = k(1)(1)(1)$$

$$\Rightarrow 8 = k$$

$$\therefore \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$$

3. Solve $\begin{vmatrix} \mathbf{x+a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} & \mathbf{x+b} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{x+c} \end{vmatrix} = \mathbf{0}$.

Solution :

$$\text{Let } A = \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} \quad \text{... (1)}$$

Putting $x = 0$ in (1) we get,

$$A = \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0 [\because R_1 \equiv R_2 \equiv R_3]$$

$\therefore (x - 0)^2 = x^2$ is a factor of A.

Putting $x = -(a + b + c)$ in (1) we get,

$$A = \begin{vmatrix} -b-c & b & c \\ a & -a-c & c \\ a & b & -a-b \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b & c \\ 0 & -a-c & c \\ 0 & b & -a-b \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$= 0$

$\therefore x + (a+b+c)$ is a factor of A.

Since the leading diagonal of A is of degree 3, only 3 factors are available and there may exist a constant k.

$$\therefore \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = k(x^2)(x+a+b+c)$$

Putting $x = -a$ we get,

$$\begin{vmatrix} 0 & b & c \\ a & -a+b & c \\ a & b & -a+c \end{vmatrix} = k(a^2)(-a+a+b+c)$$

Expanding along R_1 we get,

$-b[(-a^2) + (ab - ab)] + c(ab + a^2 - ab) = k(a^2)(b+c)$

$\Rightarrow \quad a^2b + a^2c = k(a^2)(b+c)$

$\Rightarrow \quad a^2(b+c) = k(a^2)(b+c)$

$\Rightarrow \quad k = 1$

$\therefore 1(x^2)(x+a+b+c) = 0$

$\Rightarrow \quad x = 0$ or $x = -(a+b+c)$

Hence, the values of x are 0, 0, $-(a+b+c)$.

4. Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)\times$ **$(c-a)$.**

Solution : Let $\Delta = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$ ... (1)

Putting $a = b$ we get,

$$\Delta = \begin{vmatrix} b+c & b & b^2 \\ c+b & b & b^2 \\ 2b & c & c^2 \end{vmatrix} = 0$$

[$\because R_1 \equiv R_2$]

$\Rightarrow (a-b)$ is a factor of Δ.

Putting $b = c$ in (1) we get,

$$\Delta = \begin{vmatrix} 2c & a & a^2 \\ c+a & c & c^2 \\ a+c & c & c^2 \end{vmatrix} = 0$$

[$\because R_2 \equiv R_3$]

$\Rightarrow (b-c)$ is a factor of Δ.

Putting $c = a$ in (1) we get,

$$\Delta = \begin{vmatrix} b+a & a & a^2 \\ 2a & b & b^2 \\ a+b & a & a^2 \end{vmatrix} = 0$$

[$\because R_1 \equiv R_3$]

$\Rightarrow (c-a)$ is a factor of Δ.

Putting $a = -(b+c)$ in (1) we get,

$$\Delta = \begin{vmatrix} b+c & -b-c & (-b-c)^2 \\ c-b-c & b & b^2 \\ -b-c+b & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} b+c & -(b+c) & (b+c)^2 \\ -b & b & b^2 \\ -c & c & c^2 \end{vmatrix}$$

$= 0 \quad [\because R_2 \propto R_3]$

$\therefore (a+b+c)$ is a factor of Δ.

Since the leading diagonal of Δ is of degree 4, only 4 factors and a constant k are available.

$$\therefore \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = k(a+b+c)(a-b)(b-c)(c-a)$$

Putting $a = 2, b = 1, c = 0$ we get,

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 0 & 0 \end{vmatrix} = k(3)(1)(1)(-2)$$

Expanding along R_3 we get,

$\Rightarrow \quad 3\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} = -6k$

$\Rightarrow \quad 3(2-4) = -6k$

$\Rightarrow \quad -6 = -6k$

$\Rightarrow \quad k = 1$

$$\therefore \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$$

Hence proved.

5. Solve $\begin{vmatrix} \mathbf{4-x} & \mathbf{4+x} & \mathbf{4+x} \\ \mathbf{4+x} & \mathbf{4-x} & \mathbf{4+x} \\ \mathbf{4+x} & \mathbf{4+x} & \mathbf{4-x} \end{vmatrix} = \mathbf{0}$**.**

Solution : Let $A = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix}$ ... (1)

Putting $x = 0$ in (1) we get,

$$A = \begin{vmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0 \quad [\because R_1 \equiv R_2 \equiv R_3]$$

$\Rightarrow x^2$ is a factor of (1)

Putting $x = -12$ we get,

$$A = \begin{vmatrix} 4+12 & 4-12 & 4-12 \\ 4-12 & 4+12 & 4-12 \\ 4-12 & 4-12 & 4+12 \end{vmatrix}$$

$$= \begin{vmatrix} 16 & -8 & -8 \\ -8 & 16 & -8 \\ -8 & -8 & 16 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -8 & -8 \\ 0 & 16 & -8 \\ 0 & -8 & 16 \end{vmatrix} = 0$$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$\therefore (x+12)$ is also a factor of (1).

Since the leading diagonal of A is of degree 3, only 3 factors and a constant k are available

$$\therefore A = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = k(x^2)(x+12)$$

Putting $x = 1$, we get

$$\begin{vmatrix} 3 & 5 & 5 \\ 5 & 3 & 5 \\ 5 & 5 & 3 \end{vmatrix} = k(1)^2(1+12)$$

$\Rightarrow 3(9-25) - 5(15-25) + 5(25-15) = 13k$

[$\because$ Expanding along R_1]

$\Rightarrow 3(-16) - 5(-10) + 5(10) = 13k$

$\Rightarrow -48 + 50 + 50 = 13k$

$\Rightarrow 52 = 13k$

$\Rightarrow k = 4$

$$\therefore \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 4(x^2)(x+12)$$

$\Rightarrow 4(x^2)(x+12) = 0$

$\Rightarrow x = 0$ or $x = -12$

6. Show that $\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \\ \boldsymbol{x^2} & \boldsymbol{y^2} & \boldsymbol{z^2} \end{vmatrix} = \boldsymbol{(x-y)(y-z)(z-x)}$

Solution : Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$ ... (1)

Putting $x = y$ in (1) we get,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ y & y & z \\ y^2 & y^2 & z^2 \end{vmatrix} = 0$$

[$\because C_1 \equiv C_2$]

$\therefore (x-y)$ is a factor of (1).

Putting $y = z$ in (1) we get,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & z & z \\ 1 & z^2 & z^2 \end{vmatrix} = 0 \quad [\because C_2 \equiv C_3]$$

$\therefore (y-z)$ is a factor of (1)

Putting $z = x$ in (1) we get,

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & x \\ x^2 & y^2 & x^2 \end{vmatrix} = 0 \quad [\because C_1 \equiv C_3]$$

$\therefore (z-x)$ is a factor of (1)

Since the leading diagonal of Δ is of degree 3, there are 3 factors and a constant k.

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = k(x-y)(y-z)(z-x)$$

Putting $x = 0, y = 1, z = -1$ in the above equation we get,

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = k(-1)(2)(-1)$$

$$\Rightarrow \quad 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2k \text{ [Expanded along } C_1]$$

$$1(1+1) = 2k \Rightarrow \not{2} = \not{2}k \Rightarrow k = 1$$

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

Exercise 7.4

1. Find the area of the triangle whose vertices are (0, 0), (1, 2) and (4, 3).

Solution : Let the vertices of the triangle be A(0, 0) B (1, 2) C(4, 3)

$$\text{Area of the } \Delta ABC = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \text{absolute value of } \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \text{absolute value of } \frac{1}{2}\left[0+0+1\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix}\right]$$

[Expanded along R_1]

$$= \text{absolute value of } \frac{1}{2}[3-8]$$

$$= \text{absolute value of } \frac{1}{2}[-5]$$

$$= \text{absolute value of } (-2.5)$$

$$= 2.5 \text{ Sq.units}$$

2. If (*k*, 2), (2, 4) and (3, 2) are vertices of the triangle of area 4 square units then determine the value of *k*.

Solution : Let the vertices of the triangle be A(*k*, 2) B(2, 4) and C(3, 2)

Also area of ΔABC = 4 sq. units.

We know that, area of ΔABC

$$= \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 4 = \text{absolute value of } \frac{1}{2}\begin{vmatrix} k & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 4 = \text{absolute value of } \frac{1}{2}[k(4-2) - 2(2-3) + 1(4-12)] \quad \text{[Expanded along } R_1]$$

$$\Rightarrow \quad 4 = \text{absolute value of } \frac{1}{2}[2k+2-8]$$

$$\Rightarrow \quad 4 = \text{absolute value of } \frac{1}{2}[2k-6]$$

$$\Rightarrow \quad 4 = \pm\frac{1}{2}(2k-4)$$

Case (i)

$$\text{when } 4 = \frac{1}{2}(2k-6)$$

$$\Rightarrow \quad 8 = 2k-6$$

$$\Rightarrow \quad 14 = 2k$$

$$\Rightarrow \quad k = 7$$

Case (ii)

$$\text{when } 4 = -\frac{1}{2}(2k-6)$$

$$\Rightarrow \quad 8 = -2k+6$$

$$\Rightarrow \quad 8-6 = -2k$$

$$\Rightarrow \quad 2 = -2k$$

$$\Rightarrow \quad k = -1$$

∴ The values of *k* are –1 or 7.

3. Identify the singular and non-singular matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

Solution : (i) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$|A| = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

[Expanded along R_1]

$$= 1(45-48) - 2(36-42) + 3(32-35)$$

$$= -3 - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9 = -\not{12} + \not{12} = 0$$

Since $|A| = 0$, the given matrix is singular.

(ii) Let B $= \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$

$|B| = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} + 3\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$ [Expanded along R_1]

$= 2(0-20) + 3(-42-4) + 5(30-0)$

$= -40 + 3(-46) + 5(30)$

$= -40 - 138 + 150 = -28 \neq 0$

Since $|B| \neq 0$, the given matrix is non-singular.

(iii) Let C $= \begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

$|C| = 0 - (a-b)\begin{vmatrix} b-a & 5 \\ -k & 0 \end{vmatrix} +$

$k \times \begin{vmatrix} b-a & 0 \\ -k & -5 \end{vmatrix} = -(a-b)(5k) + k[-5(b-a)]$

$= -(a-b)(5k) + 5k(a-b) = 0$

Since $|C| = 0$, the given matrix is singular.

4. Determine the values of a and b so that the following matrices are singular:

(i) $A = \begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$ (ii) $B = \begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

Solution :

(i) Given A $= \begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$

Since A is a singular matrix

$|A| = 0$

$\begin{vmatrix} 7 & 3 \\ -2 & a \end{vmatrix} = 0$

$\Rightarrow 7a + 6 = 0$

$\Rightarrow 7a = -6$

$\Rightarrow a = -\frac{6}{7}$

(ii) B $= \begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

Since B is a singular matrix, $|B| = 0$

$\therefore$ If $a = -\frac{6}{7}$ then $\begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$ is singular.

$|B| = \begin{vmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0$

$= (b-1)\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = 0$

[Expanded along R_1]

$\Rightarrow (b-1)(4+4) - 2(12-2) + 3(-6-1) = 0$

$(b-1)(8) - 2(10) + 3(-7) = 0$

$8b - 8 - 20 - 21 = 0$

$8b - 49 = 0$

$\Rightarrow 8b = 49$

$\Rightarrow b = \frac{49}{8}$

If $b = \frac{49}{8}$ then $\begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$ is singular.

5. If $\cos 2\theta = 0$, determine $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$

Solution : Given $\cos 2\theta = 0$

Let A $= \begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$

$= \begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} \begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}$

$= \begin{Vmatrix} 0+\cos^2\theta+\sin^2\theta & 0+\sin\theta+0 & 0+0+\sin\theta\cos\theta \\ 0+\sin\theta\cos\theta+0 & \cos^2\theta+\sin^2\theta+0 & \sin\theta\cos\theta+0+0 \\ 0+0+\sin\theta\cos\theta & \sin\theta\cos\theta+0+0 & \sin^2\theta+0+\cos^2\theta \end{Vmatrix}$

$\left[\because \sin 2\theta\sqrt{1-\cos^2 2\theta} = \sqrt{1} = 1\right]$

$= \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = 1\left(1-\frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{2-\frac{1}{4}}\right) + \frac{1}{2}\left(\frac{1}{4}-\frac{1}{2}\right)$

$= \frac{3}{4} - \frac{1}{8} - \frac{1}{8} = \frac{6-1-1}{8} = \frac{4}{8} = \frac{1}{2}$

6. Find the value of the product:

$\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$

Solution : $\begin{vmatrix}\log_3 64 & \log_4 3\\ \log_3 8 & \log_4 9\end{vmatrix} \times \begin{vmatrix}\log_2 3 & \log_8 3\\ \log_3 4 & \log_3 4\end{vmatrix}$

$$= \begin{vmatrix}\log_3 64\cdot\log_2 3+\log_4 3\cdot\log_3 4 & \log_3 64\cdot\log_8 3+\log_4 3\cdot\log_3 4\\ \log_3 8\cdot\log_2 3+\log_4 9\cdot\log_3 4 & \log_3 8\cdot\log_8 3+\log_4 9\cdot\log_3 4\end{vmatrix}$$

$$= \begin{vmatrix}6\log_3 2\log_2 3+1 & 2\log_3 8\log_8 3+1\\ 3\log_3 2\log_2 3+1 & 1+2\log_4 3\log_3 4\end{vmatrix}$$

$$= \begin{vmatrix}6+1 & 2+1\\ 3+2 & 1+2\end{vmatrix} = \begin{vmatrix}7 & 3\\ 5 & 3\end{vmatrix} = 21-15=6$$

EXERCISE 7.5

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER FROM THE GIVEN FOUR ALTERNATIVES.

1. If $a_{ij} = \frac{1}{2}(3i-2j)$ and $A = [a_{ij}]_{2\times 2}$ is

(1) $\begin{bmatrix}\frac{1}{2} & 2\\ -\frac{1}{2} & 1\end{bmatrix}$ (2) $\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 2 & 1\end{bmatrix}$

(3) $\begin{bmatrix}2 & 2\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}$ (4) $\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ 1 & 2\end{bmatrix}$

Hint : $a_{ij} = \frac{1}{2}(3i-2j)$

$a_{11} = \frac{1}{2}(3-2) = \frac{1}{2}$, $a_{12} = \frac{1}{2}(3-4) = -\frac{1}{2}$

$a_{21} = \frac{1}{2}(6-2) = 2$, $a_{22} = \frac{1}{2}(6-4) = \frac{2}{2} = 1$

$\therefore A = \begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 2 & 1\end{bmatrix}$ **[Ans:** (2) $\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 2 & 1\end{bmatrix}$**]**

2. What must be the matrix X, if $2X+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}3 & 8\\ 7 & 2\end{bmatrix}$?

(1) $\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$ (2) $\begin{bmatrix}1 & -3\\ 2 & -1\end{bmatrix}$

(3) $\begin{bmatrix}2 & 6\\ 4 & -2\end{bmatrix}$ (4) $\begin{bmatrix}2 & -6\\ 4 & -2\end{bmatrix}$

Hint : $2X+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}3 & 8\\ 7 & 2\end{bmatrix}$

$\Rightarrow 2X = \begin{bmatrix}3 & 8\\ 7 & 2\end{bmatrix}-\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}2 & 6\\ 4 & -2\end{bmatrix}$

$X = \begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$

[Ans: (1) $\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$**]**

3. Which one of the following is true about the matrix $\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 5\end{bmatrix}$?

(1) a scalar matrix
(2) a diagonal matrix
(3) an upper triangular matrix
(4) a lower triangular matrix

[Ans: (2) a diagonal matrix**]**

4. If A and B are two matrices such that A + B and AB are both defined, then

(1) A and B are two matrices not necessarily of same order
(2) A and B are square matrices of same order
(3) Number of columns of A is equal to the number of rows of B
(4) A = B.

Hint : For addition both A and B must be of same order to get AB, number of columns of A should be equal to number of rows of B.

If A and B ae square matrices of same order both condition are satisfied.

[Ans: (2) A and B are square matrices of same order]

5. If $A=\begin{bmatrix}\lambda & 1\\ -1 & -\lambda\end{bmatrix}$, then for what value of λ^2, $A^2 = 0$?

(1) 0 (2) ±1 (3) −1 (4) 1

Hint : $A = \begin{bmatrix}\lambda & 1\\ -1 & -\lambda\end{bmatrix}$

$A^2 = 0$

$\Rightarrow \begin{bmatrix}\lambda & 1\\ -1 & -\lambda\end{bmatrix}\begin{bmatrix}\lambda & 1\\ -1 & -\lambda\end{bmatrix} = \begin{bmatrix}\lambda^2-1 & 0\\ 0 & -1+\lambda^2\end{bmatrix} = 0$

$\Rightarrow \quad \lambda^2 - 1 = 0$

$\Rightarrow \quad \lambda^2 = 1$

$\lambda = \pm 1$ **[Ans:** (2) ± 1**]**

6. If A = $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, then the values of a and b are

(1) $a = 4, b = 1$ (2) $a = 1, b = 4$
(3) $a = 0, b = 4$ (4) $a = 2, b = 4$

Hint : $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$

$$A + B = \begin{bmatrix} a+1 & 0 \\ b+2 & -2 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} a+1 & 0 \\ b+2 & -2 \end{bmatrix}\begin{bmatrix} a+1 & 0 \\ b+2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (a+1)^2 & 0 \\ (a+1)(b+2)-2(b+2) & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

Since $(A+B)^2 = A^2 + B^2$

$$\begin{bmatrix} (a+1)^2 & 0 \\ (a+1)(b+2)-2(b+2) & 4 \end{bmatrix} = \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

$a - 1 = 0$

$\Rightarrow \quad a = 1, b = 4$ **[Ans:** (2) $a = 1, b = 4$**]**

7. If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to

(1) (2, –1) (2) (–2, 1)
(3) (2, 1) (4) (–2, –1)

Hint : Given $AA^T = 9I$

$\Rightarrow \quad |AA^T| = |9I| \quad [\because |A| = |A^T|]$

$|A|.|A| = 9^3|I|$

$|A|^2 = 9^3 = 9 \times 9 \times 9 = 9^2 \times 3^2 = 27^2$

$|A| = 27$

$$\therefore \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{vmatrix} = 27$$

$\Rightarrow \quad 1(b + 4) - 2(2b + 2a) + 2(4 - a) = 27$

$\Rightarrow \quad b + 4 - 4b - 4a + 8 - 2a = 27$

$\Rightarrow \quad -6a - 3b + 12 = 27$

Only (–2, –1) satisfies this equation.

[Ans: (4) (–2, –1)**]**

8. If A is a square matrix, then which of the following is not symmetric?

(1) $A + A^T$ (2) AA^T (3) A^TA (4) $A - A^T$

Hint : $(A-A^T)^T = A^T - (A^T)^T$

$= A^T - A = -(A - A^T)$

[Ans: (4) $A - A^T$**]**

9. If A and B are symmetric matrices of order n, where $(A \neq B)$, then

(1) A + B is skew-symmetric
(2) A + B is symmetric
(3) A + B is a diagonal matrix
(4) A + B is a zero matrix

Hint : $(A+B)^T = A^T + B^T = A + B$

[Ans: (2) A + B is symmetric**]**

10. If A = $\begin{bmatrix} a & x \\ y & a \end{bmatrix}$ and if $xy = 1$, then det (AA^T) is equal to

(1) $(a - 1)^2$ (2) $(a^2 + 1)^2$
(3) $a^2 - 1$ (4) $(a^2 - 1)^2$

Hint : $A = \begin{bmatrix} a & x \\ y & a \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & y \\ x & a \end{bmatrix}$

det (AA^T) = det (A) . det (A^T) = det (A). det (A)

det (A) $= \begin{vmatrix} a & x \\ y & a \end{vmatrix} = a^2 - xy = a^2 - 1 \quad [\because xy = 1]$

det $(AA^T) = (a^2 - 1)(a^2 - 1) = (a^2 - 1)^2$

[Ans: (4) $(a^2 - 1)^2$**]**

11. The value of x, for which the matrix A = $\begin{bmatrix} e^{x-2} & e^{7+x} \\ e^{2+x} & e^{2x+3} \end{bmatrix}$ is singular is **[March - 2019]**

(1) 9 (2) 8 (3) 7 (4) 6

Hint : Since A is singular, $|A| = 0$

$$\therefore \begin{vmatrix} e^{x-2} & e^{7+x} \\ e^{2+x} & e^{2x+3} \end{vmatrix} = 0$$

$$e^{x-2}.e^{2x+3} - e^{2+x}.e^{7+x} = 0$$

$$e^{3x+1} - e^{9+2x} = 0$$
$$\Rightarrow e^{3x+1} = e^{9+x}$$
$$3x+1 = 9+2x$$
$$\Rightarrow x = 8$$ **[Ans: (2) 8]**

12. If the points (x, –2), (5, 2), (8,8) are collinear, then x is equal to **[Hy- 2018]**

(1) –3 (2) $\frac{1}{3}$ (3) 1 (4) 3

Hint : Since the points are collinear, area of the triangle is 0.

Absolute value of $\frac{1}{2}\begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$

Absolute value of $\frac{1}{2}\left[x\begin{vmatrix} 2 & 1 \\ 8 & 1 \end{vmatrix} + 2\begin{vmatrix} 5 & 1 \\ 8 & 1 \end{vmatrix} + 1\begin{vmatrix} 5 & 2 \\ 8 & 8 \end{vmatrix}\right] = 0$

Absolute value of $\frac{1}{2}$ $[x(2-8) + 2(5-8) + 1(40-16)] = 0$

Absolute value of $\frac{1}{2}[-6x - 6 + 24] = 0$

Absolute value of $\frac{1}{2}[-6x + 18] = 0$

Absolute value of $-6x + 18 = 0 \Rightarrow 6x = 18 \Rightarrow x = 3$

[Ans: (4) 3]

13. If $\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2} \neq 0$, then the area of the triangle whose vertices are $\left(\frac{x_1}{a}, \frac{y_1}{a}\right), \left(\frac{x_2}{b}, \frac{y_2}{b}\right), \left(\frac{x_3}{c}, \frac{y_3}{c}\right)$ is

(1) $\frac{1}{4}$ (2) $\frac{1}{4}abc$
(3) $\frac{1}{8}$ (4) $\frac{1}{8}abc$

Hint : Area of the triangle

$$= \text{Absolute value of } \frac{1}{2}\begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{a} & 1 \\ \frac{x_2}{b} & \frac{y_2}{b} & 1 \\ \frac{x_3}{c} & \frac{y_3}{c} & 1 \end{vmatrix}$$

Consider $\frac{1}{2}\begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{a} & 1 \\ \frac{x_2}{b} & \frac{y_2}{b} & 1 \\ \frac{x_3}{c} & \frac{y_3}{c} & 1 \end{vmatrix}$

$R_1 \times a$, $R_2 \times b$, $R_3 \times c$ and divide by abc

$$= \frac{1}{2abc}\begin{vmatrix} x_1 & y_1 & a \\ x_2 & y_2 & b \\ x_3 & y_3 & c \end{vmatrix} = \frac{1}{2\cancel{abc}} \times \frac{\cancel{abc}}{4} = \frac{1}{8}$$

$$\left[\because \begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2} \Rightarrow \begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix} = \frac{abc}{4}\right]$$

[Ans: (3) $\frac{1}{8}$]

14. If the square of the matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is the unit matrix of order 2, then α, β and γ should satisfy the relation.

(1) $1 + \alpha^2 + \beta\gamma = 0$ (2) $1 - \alpha^2 - \beta\gamma = 0$
(3) $1 - \alpha^2 + \beta\gamma = 0$ (4) $1 + \alpha^2 - \beta\gamma = 0$

Hint : Given $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} \alpha^2 + \beta\gamma & \cancel{\alpha\beta} - \cancel{\alpha\beta} \\ \cancel{\alpha\gamma} - \cancel{\alpha\gamma} & \beta\gamma + \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \beta\gamma + \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \alpha^2 + \beta\gamma = 1$$
$$\Rightarrow 1 - \alpha^2 - \beta\gamma = 0$$

[Ans: (2) $1 - \alpha^2 - \beta\gamma = 0$]

15. If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$ is

(1) Δ (2) $k\Delta$ (3) $3k\Delta$ (4) $k^3\Delta$

Hint : Taking k common form R_1, R_2 and R_3 we get,

$$\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix} = k^3\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = k^3\Delta$$ **[Ans: (4) $k^3\Delta$]**

16. A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

(1) 6 (2) 3 (3) 0 (4) –6

Hint : $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$

$$\begin{vmatrix} -x & -x & -x \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$$

[∵ Applying $R_1 \to R_1 + R_2 + R_3$]

$$-x\begin{vmatrix} 1 & 1 & 1 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$$

$\Rightarrow$ $x = 0$

is a root of the equation. **[Ans:** (3) 0**]**

17. The value of the determinant of A = $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ **is**

(1) $-2abc$ (2) abc

(3) 0 (4) $a^2 + b^2 + c^2$

Hint : $|A| = \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0 - a\begin{vmatrix} -a & c \\ b & 0 \end{vmatrix} - b\begin{vmatrix} -a & 0 \\ b & -c \end{vmatrix}$

$= -a(-bc) - b(ac) = abc - abc = 0$ **[Ans:** (3) 0**]**

18. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in geometric progression with the same common ratio, then the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are

(1) vertices of an equilateral triangle
(2) vertices of a right angled triangle
(3) vertices of a right angled isosceles triangle
(4) collinear

Hint : x_1, x_2, x_3 are in G.P.
Let it be represented as a, ar, ar^2 y_1, y_2, y_3 are in G.P
Let it be represented by b, br, br^2
(They have same common ratio)

$$\text{Area of} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_2 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ ar & br & 1 \\ ar^2 & br^2 & 1 \end{vmatrix}$$

$$= \frac{ab}{2}\begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix}$$

[Ans: (4) collinear**]**

19. If $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0, 0 \le y < 1, 1 \le z < 2$, then the value of the determinant $\begin{vmatrix} \lfloor x \rfloor+1 & \lfloor y \rfloor & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor+1 & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor & \lfloor z \rfloor+1 \end{vmatrix}$ **is**

(1) $\lfloor z \rfloor$ (2) $\lfloor y \rfloor$ (3) $\lfloor x \rfloor$ (4) $\lfloor x \rfloor + 1$

Hint : $\begin{vmatrix} \lfloor x \rfloor+1 & \lfloor y \rfloor & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor+1 & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor & \lfloor z \rfloor+1 \end{vmatrix} = \begin{vmatrix} -1+1 & 0 & 1 \\ -1 & 0+1 & 1 \\ -1 & 0 & 1+1 \end{vmatrix}$

$$\left[\begin{aligned} \because -1 \le x < 0 &\Rightarrow \lfloor x \rfloor = -1 \\ 0 \le y < 1 &\Rightarrow \lfloor y \rfloor = 0 \\ 1 \le z < 2 &\Rightarrow \lfloor z \rfloor = 1 \end{aligned}\right]$$

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$$ [Expanded along R_1]

$= 1[0 + 1] = 1 = \lfloor z \rfloor$ **[Ans:** (1) $\lfloor z \rfloor$**]**

20. If $a \ne b$, b, c satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$**, then abc =**

(1) $a + b + c$ (2) 0

(3) b^3 (4) $ab + bc$

Hint : $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$ and $a \ne b$

$\Rightarrow$ $a(b^2 - ac) - 2b(3b - 4c) + 2c(3a - 4b) = 0$

$\Rightarrow$ $ab^2 - a^2c - 6b^2 + \cancel{8bc} + 6ac - \cancel{8bc} = 0$

$\Rightarrow$ $ab^2 - 6b^2 - a^2c + 6ac = 0 \Rightarrow b^2(a - 6) - ac(a - 6) = 0$

$\Rightarrow$ $(a - 6)(b^2 - ac) = 0 \Rightarrow a = 6$ or $b^2 = ac$

$\Rightarrow$ $b^2 = ac \Rightarrow b^3 = abc$ **[Ans:** (3) b^3**]**

21. If A = $\begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ **and B =** $\begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$**, then B is given by**

(1) B = 4A (2) B = –4A

(3) B = –A (4) B = 6A

Hint : $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ and

$$B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$$

$$\text{Then } B = -\begin{vmatrix} -2 & 4 & 8 \\ 6 & 2 & 0 \\ -2 & 4 & 2 \end{vmatrix} R_1 \leftrightarrow R_3$$

Taking out 2 from R_1 and 2 from R_2, we get

$$B = -(2)(2)\begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix} = -4A$$

[Ans: (2) B = – 4A**]**

22. If A is skew-symmetric of order n and C is a column matrix of order $n \times 1$, then C^TAC is

(1) an identity matrix of order n
(2) an identity matrix of order 1
(3) a zero matrix of order 1
(4) an identity matrix of order 2

Hint : C is of order $n \times 1 \Rightarrow C^T$ is of order $1 \times n$

$\therefore C^TA$ of order $1 \times n$

And C^TAC is of order $(1 \times \not{n}) \times (\not{n} \times 1) = (1 \times 1)$

Since A is a skew-symmetric matrix, C^TAC is a zero matrix of order 1.

[Ans: (3) a zero matrix of order 1**]**

23. The matrix A satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ A = $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is

(1) $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -4 \\ 1 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

Hint :

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+3c & b+3d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow c = 0$ and $d = -1$

Also $a + 3c = 1 \Rightarrow a + 0 = 1 \Rightarrow a = 1$

$b + 3d = 1 \Rightarrow b + 3(-1) = 1 \Rightarrow b = 4$

$$\therefore A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$$

[Ans: (3) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$**]**

24. If A + I = $\begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, then (A + I) (A – I) is equal to

(1) $\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -5 & 4 \\ -8 & 9 \end{bmatrix}$

(3) $\begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$ (4) $\begin{bmatrix} -5 & -4 \\ -8 & -9 \end{bmatrix}$

Hint :

$$A + I = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$$

$$A + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 2 & -2 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$$

$$\therefore (A + I)(A - I) = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-8 & -6+2 \\ 4+4 & -8-1 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$$

[Ans: (1) $\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$**]**

25. Let A and B be two symmetric matrices of same order. Then which one of the following statement is not true?

(1) A + B is a symmetric matrix
(2) AB is a symmetric matrix
(3) $AB = (BA)^T$
(4) $A^T B = AB^T$

Hint : For symmetric matrix $= A^T = A$

$(BA)^T = A^T B^T = AB$

$A^T B = AB = AB^T$

Sum of two symmetric matrix is also a symmetric matrix.

AB is a symmetric matrix is not true.

[Ans: (2) AB is a symmetric matrix**]**

ADDITIONAL PROBLEMS
SECTION - A (1 MARK)

1. **If $\begin{bmatrix} 4 & 3 \\ -2 & x \end{bmatrix}$ is singular then the value of x is** **[Hy - 2018]**

(1) $\frac{3}{2}$ (2) $-\frac{3}{2}$ (3) 3 (4) -2

[Ans: (2) $-\frac{3}{2}$]

2. **Which one of the following is not true about the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?** **[March - 2019]**

(1) an upper triangular matrix
(2) a lower triangular matrix
(3) a scalar matrix
(4) a diagonal matrix **[Ans: (3) a scalar matrix]**

3. **If $\begin{pmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{pmatrix}$ is a singular matrix, then λ is**

(1) $\lambda = 2$ (2) $\lambda \neq 2$ (3) $\lambda = \frac{-8}{5}$ (4) $\lambda \neq \frac{-8}{5}$

Hint : Expanding along C_1, we get

$2(6-5)+1(5\lambda+6) = 0$

$\lambda = \frac{-8}{5}$ **[Ans: (3) $\lambda = \frac{-8}{5}$]**

4. **If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$ then**

(1) $f(a) = 0$ (2) $f(b) = 0$
(3) $f(0) = 0$ (4) $f(1) = 0$

Hint : $f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = a(bc) - b(ac) = 0$

[Ans: (4) $f(0) = 0$]

5. **Find the odd one out of the following :**

(1) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 0 & \frac{-7}{2} \\ \frac{7}{2} & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 0 & 3.2 \\ -3.2 & 0 \end{bmatrix}$ (4) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Hint : (1), (2), (3) are skew symmetric and (4) symmetric

[Ans: (4) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$]

6. **Choose the correct statement**

(1) Matrix addition is not associative
(2) Matrix addition is not commutative
(3) Matrix multiplication is associative
(4) Matrix multiplication is commutative

[Ans: (3) Matrix multiplication is associative]

7. **Choose the incorrect statement**

(1) Matrix multiplicationis non commutative
(2) Matrix addition is associative
(3) Singular matrices have inverse
(4) Non singular matrices have inverse

[Ans: (3) Singular matrices have inverse]

8. **Assertion (A) :** The inverse of $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$ does not exist.

Reason (R) : The matrix is non-singular

(1) Both A and R are true and R is the correct explanation of A
(2) Both A and R are true and R is not a correct explantion of A
(3) A is true but R is false
(4) A is false but R is true

[Ans: (3) A is true but R is false]

SECTION - B (2 MARKS)

1. **Find x, y, z and w such that $\begin{bmatrix} x-y & 2z+w \\ 2x-y & 2x+w \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$**

Solution : Given $\begin{bmatrix} x-y & 2z+w \\ 2x-y & 2x+w \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$

Equating the corresponding entries on both sides, we get

$x - y = 5$ ... (1)

(–) (+) (–)

$2x - y = 12$ ... (2)

$-x = -7$

$\Rightarrow x = 7$

Substituting $x = 7$ in (1) we get,

$7 - y = 5$

$\Rightarrow y = 2$

$2z + w = 3$ ... (3)

$2x + w = 15$ ... (4)

$2(7) + w = 15$

$\Rightarrow 14 + w = 15$

$\Rightarrow \quad w = 1$

Substituting $w = 1$ in (3) we get,

$2z + 1 = 3$

$\Rightarrow \quad 2z = 2 \Rightarrow z = 1$

$\therefore x = 7, y = 2, z = 1$ and $w = 1$

2. For what value of x the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular.

Solution : The matrix A is singular if $|A| = 0$

$$\Rightarrow \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix} = 0$$

$$\Rightarrow (-6-2) + 2(-3-x) + 3(2-2x) = 0$$

$$\Rightarrow -8 - 6 - 2x + 6 - 6x = 0$$

$$\Rightarrow -8x = 8 \Rightarrow x = -1$$

3. Without expanding evaluate the determinant $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

Solution : Let $\Delta = \begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

Applying $C_1 \to C_1 + (-8) C_3$ we get

$$\Delta = \begin{vmatrix} 1 & 1 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix} = 0 \quad [\because C_1 \equiv C_2]$$

$$\Rightarrow \Delta = 0$$

SECTION - C (3 MARKS)

1. Prove that square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix. **[March - 2019]**

Solution : Let A be square matrix

Then $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

We know that $A + A^T$ is symmetric

$A - A^T$ is skew symmetric

$\therefore$ A can be written as sum of symmetric and skew symmetric matrix.

1. Find X and Y if $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

Solution :

$$X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \text{... (1)}$$

$$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{... (2)}$$

$$\Rightarrow 2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Substituting $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ we get,

$$\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$$

$$\Rightarrow Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

2. Find non-zero values of x satisfying the matrix equation, $x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2+8 & 24 \\ 10 & 6x \end{bmatrix}$

Solution : Given $x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2+8 & 24 \\ 10 & 6x \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x^2 & 2x \\ 3x & x^2 \end{bmatrix} + \begin{bmatrix} 16 & 10x \\ 8 & 8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2+16 & 2x+10x \\ 3x+8 & x^2+8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2+16 & 12x \\ 3x+8 & x^2+8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$12x = 48 \quad \Rightarrow x = 4$

and $x^2 + 8x = 12x \quad \Rightarrow x^2 - 4x = 0$

$\Rightarrow \quad x(x-4) = 0 \quad \Rightarrow x = 0, 4$

Since $x = 0$ is not possible

$\Rightarrow \quad x = 4.$

3. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ find the values of α for which $A^2 = B$.

Solution : Given $A^2 = B$

$$\Rightarrow \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2+0 & 0+0 \\ \alpha+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow \quad \alpha^2 = 1$ or $\alpha + 1 = 5$

$\Rightarrow \quad \alpha = \pm 1$ or $\alpha = 4$

which is not possible.

Hence, there is no value of α for which $A^2 = B$ is true.

4. Show that the points $(a, b + c)$ $(b, c + a)$ and $(c, a + b)$ are collinear.

Solution : Let the points be $A(a, b + c)$, $B(b, c + a)$ and $C(c, a + b)$

$$\text{Area of the } \Delta ABC = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \text{absolute value of } \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$ we get,

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix}$$

[Taking out $(a + b + c)$ common from C_1]

$$= \frac{1}{2}(a+b+c)(0) = 0$$

$\therefore$ Since area of $\Delta ABC = 0$, the given points are collinear.

SECTION - D (5 MARKS)

1. Using factor theorem, show that **[Hy - 2018]**

$$\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$$

Solution :

$$A = \begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

Let $a = -b$

$$\begin{vmatrix} 2b & 0 & c-b \\ 0 & -2b & b+c \\ c-b & c+b & -2c \end{vmatrix} = \begin{vmatrix} 2b & 0 & c-b \\ 2b & -2b & 2c \\ b+c & l+c & -(b+c) \end{vmatrix}$$

$R_2 \to R_2 + R_1$
$R_3 \to R_3 + R_1$

$$= 2b(b+c)\begin{vmatrix} 2b & 0 & c-b \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$\Rightarrow (a + b)$ is a factor.

Similarly $(b + c)$ and $(c + a)$ are factors

($\therefore$ $|A|$ is in cyclic symmetric form in a, b, c)

Degree of $|A|$ is 3.

The degree of the obtained factor is 3.

$\therefore$ $|A| = k(a + b)(b + c)(c + a)$

Substituting values, we get $k = 4$

$$\therefore \begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$$

2. Without expanding evaluate the determinant $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

Solution :

$$\text{Let } \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta+\cos\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\cos\delta+\cos\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta+\cos\gamma\sin\delta \end{vmatrix}$$

[$\because$ $\sin(A + B) = \sin A \cos B + \cos A \sin B$]

Applying $C_3 \to C_3 - (\cos\delta)C_1 - (\sin\delta)C_2$ we get,

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$$

Expanding along C_3, we get

$$\Delta = 0$$

3. Show that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$

Solution :

$$\text{Let } \Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Taking x, y, z common from C_1, C_2 and C_3 respectively,

$$\Delta = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ we get,

$$\Delta = xyz \begin{vmatrix} 1 & 0 & 1 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

Taking $(y-x)$ and $(z-x)$ common from C_2 & C_3 respectively

$$\Delta = xyz(y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

Expanding along R_1 we get,

$\Delta = xyz\,(y-x)\,(z-x)\,[z+\not{x}-y-\not{x}]$

$\Delta = xyz\,(y-x)\,(z-x)\,[z-y]$

$\Delta = xyz\,(x-y)\,(y-z)\,(z-x)$

Hence proved.

POINTS TO REMEMBER

In this chapter we have acquired the knowledge of

- A matrix is a rectangular array of real numbers or real functions on $\mathbb{R}$ or complex numbers.
- A matrix having m rows and n columns, then the order of the matrix is $m \times n$.
- A matrix A $[a_{ij}]_{m \times n}$ is said to be a
  square matrix if $m = n$
  row matrix if $m = 1$
  column matrix if $n = 1$
  zero matrix if $a_{ij} = 0\ \forall\ i$ and j
  diagonal matrix if $m = n$ and $a_{ij} = 0\ \forall\ i \neq j$
  scalar matrix if $m = n$ and $a_{ij} = 0\ \forall\ i \neq j$ and $a_{ii} = \lambda$ for all i
  unit matrix or identity matrix if $m = n$ and $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1\ \forall\ i$
  upper triangular matrix if $m = n$ and $a_{ij} = 0\ \forall\ i > j$
  lower triangular matrix if $m = n$ and $a_{ij} = 0\ \forall\ i < j$.
- Matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are said to be equal if $a_{ij} = b_{ij}\ \forall\ i$ and j
- If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then $A + B = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$
- If $A = [a_{ij}]_{m\times n}$ and λ is a scalar, then $\lambda A = [\lambda a_{ij}]_{m\times n}$
- $-A = (-1)A$
- $A + B = B + A$
- $A - B = A + (-1)B$
- $(A + B) + C = A + (B + C)$ where A, B and C have the same order.
- $A(BC) = (AB)C$ (ii) $A(B+C) = AB+AC$ (iii) $(A+B)C = AC+BC$
- The transpose of A, denoted by A^T is obtained by interchanging rows and columns of A.
  (i) $(A^T)^T = A$, (ii) $(kA)^T = kA^T$, (iii) $(A + B)^T = A^T + B^T$, (iv) $(AB)^T = B^TA^T$
- A square matrix A is called
  (i) symmetric if $A^T = A$ and (ii) skew-symmetric if $A^T = -A$

- Any square matrix can be expressed as sum of a symmetric and skew-symmetric matrices.
- The diagonal entries of a skew-symmetric must be zero.
- For any square matrix A with real entries, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric and further $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.
- Determinant is defined only for square matrices.
- $|A^T| = |A|$.
- $|AB| = |A|\,|B|$ where A and B are square matrices of same order.
- If $A = [a_{ij}]_{m \times n}$, then $|kA| = k^n\,|A|$, where k is a scalar.
- A determinant of a square matrix A is the sum of products of elements of any row (or column) with its corresponding cofactors; for instance, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.
- If the elements of a row or column is multiplied by the cofactors of another row or column, then their sum is zero; for example, $a_{11}A_{13} + a_{12}A_{23} + a_{13}A_{33} = 0$.
- The determinant value remains unchanged if all its rows are interchanged by its columns.
- If all the elements of a row or a column are zero, then the determinant is zero.
- If any two rows or columns are interchanged, then the determinant changes its sign.
- If any two rows or columns are identical or proportional, then the determinant is zero.
- If each element of a row or a column is multiplied by constant k, then determinant gets multiplied by k.
- If each element in any row (column) is the sum of r terms, then the determinant can be expressed as the sum of r determinants.
- A determinant remains unaltered under a row (R_i) operation of the form $R_i + \alpha R_j + \beta R_k$ ($j, k \neq i$) or a Column (C_i) operation of the form $C_i + \alpha C_j + \beta C_k$ ($j, k \neq i$) where α, β are scalars.
- **Factor theorem :** If each element of $|A|$ is a polynomial in x and if $|A|$ vanishes for $x = a$, then $x - a$ is a factor of $|A|$.
- Area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
- If the area is zero, then the three points are collinear.
- A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

08 VECTOR ALGEBRA-I

MUST KNOW DEFINITIONS

- **Scalar:** A Scalar is a quantity that is determined by its magnitude.
- **Vector:** A vector is a quantity that is determined by both its magnitude and its direction and hence it is a directed line segment.
- **Position Vector:** Let O be the origin and P be any point (in the plane or space) Then the vector $\overrightarrow{OP}$ is called the position vector.
- **Magnitude of a Vector:** Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}|$ is a positive number which is a measure of its length.
- The arrow indicates the direction of the vector.
- **Types of vectors:**

1. **Zero or null vector:** A vector whose initial and terminal points are coincident.
2. **Unit Vector:** A vector whose modulus is unity.
3. **Like and unlike vectors:** Like vectors have the same sense of direction and unlike vectors have opposite directions.
4. **Co-initial vectors:** Vectors having the same initial point.
5. **Co-terminus vectors:** Vectors having the same terminal point.
6. **Collinear or parallel vectors:** Vectors having the same line of action or have the lines of action parallel to one another.
7. **Co-planar vectors:** Vectors parallel to the same plane or they lie in the same plane.
8. **Negative vector:** Vector which has the same magnitude as that of $\vec{a}$ but opposite direction is called the negative of $\vec{a}$.
9. **Reciprocal of a vector:** vector which has the same direction as that of $\vec{a}$ but has magnitude reciprocal to that of $\vec{a}$

$$|(\vec{a})^{-1}| = \frac{1}{a}$$

10. **Free and localised vector:**

    When the origin of the vector is any point it is called as a **free vector**, but when it is restricted to a certain specific point it is said to be a **localised vector.**

- **Properties of addition of vectors:**

1. Vector addition is commutative ($\vec{a}+\vec{b} = \vec{b}+\vec{a}$)
2. Vector addition is associative ($\vec{a}+\vec{b}$) + $\vec{c} = \vec{a}+(\vec{b}+\vec{c})$
3. For every $\vec{a}$, $\vec{a}+\vec{0} = \vec{0}+\vec{a}$ where $\vec{0}$ is the null vector (additive identity)
4. For every $\vec{a}$, $\vec{a}+(-\vec{a}) = (-\vec{a})+\vec{a}$ [additive Inverse]

- **Multiplication of a vector by a scalar:**

1. $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$ where $\vec{a}$ and $\vec{b}$ are any two vectors and m is a scalar.
2. $(-m)(-\vec{a}) = m\vec{a}$
3. $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$
4. $(m+n)\vec{a} = m\vec{a}+n\vec{a}$
5. $m(\vec{a}+\vec{b}) = m\vec{a}+m\vec{b}$
6. $m(\vec{a}-\vec{b}) = m\vec{a}-m\vec{b}$

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$ where $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the position vectors of A and B respectively.

- **Rectangular resolution of a vector in two dimensions:**

If P(x, y) is a point then $\overrightarrow{OP} = x\hat{i} + y\hat{j}$ where $\hat{i}$ and $\hat{j}$ are unit vectors along OX and OY respectively.

- **Rectangular resolution of a vector in three dimensions:**

If P(x, y, z) is a point in space, then $\overrightarrow{OP} = x\hat{i} + y\hat{j} + y\hat{k}$ where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along OX, OY and OZ respectively.

- **Scalar Product:** $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, $0\leq\theta\leq\pi$.

Properties of Scalar Product:

1. $\vec{a} \cdot \vec{b}$ is a real number
2. $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$
3. If $\theta = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$
4. If $\theta = \pi$, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$
5. $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ where θ is the angle between $\vec{a}$ and $\vec{b}$
6. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (Distributive property)
7. $(l\vec{a}) \cdot \vec{b} = l(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (l\vec{b})$ where λ is a scalar

Application of dot product in geometry, physics and trigonometry

1. Projection of $\vec{a}$ on other vector $\vec{b}$ is $\vec{a} \cdot \vec{b}$ (or) $\vec{a} \cdot \left(\dfrac{\vec{b}}{|\vec{b}|}\right)$ (or) $\dfrac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$
2. If $\theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ then P the projection vector will be a zero vector.
3. Work done = $\vec{F}\cdot\vec{d}$ where $\vec{F}$ is the force and $\vec{d}$ is the displacement.

Vector (cross) product of two vectors :

Vector product of two vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\, \hat{n}; 0 \leq \theta \leq \pi$. $\hat{n}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

Properties of cross product :

1. $\vec{a} \times \vec{b}$ is a vector
2. $\vec{a} \times \vec{b} = \vec{0}$ if and only if $\vec{a} \parallel \vec{b}$
3. If $\theta = \dfrac{\pi}{2}$, $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$
4. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$ and $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$
5. $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

6. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Distributing vector product over addition)

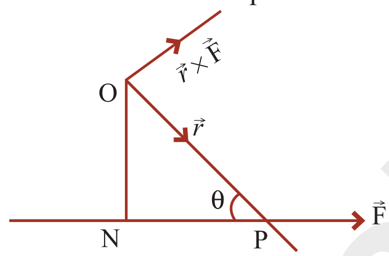
7. $l(\vec{a} \times \vec{b}) = (l\vec{a}) \times \vec{b} = \vec{a} \times (l\vec{b})$

❑ **Application of vector product in geometry, trigonometry and physics**

1. Area of a triangle = $\frac{1}{2}\left|\vec{a} \times \vec{b}\right|$ where $\vec{a}$ and $\vec{b}$ are the adjacent sides of a triangle.

2. Area of a parallelogram = $|\vec{a} \times \vec{b}|$ where $\vec{a}, \vec{b}$ represent the adjacent sides of a parallelogram.

3. Moment (or) Torque of force $\vec{F}$ about the point O is defined as $\vec{m} = \vec{r} \times \vec{F}$



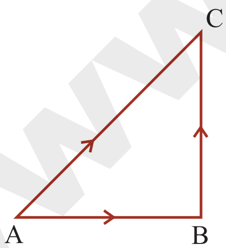
FORMULAE TO REMEMBER

❑ If P(*x, y, z*) is a point in space with respect to the origin O(0, 0, 0) then $|\vec{r}| = \overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$ (magnitude of $\overrightarrow{OP}$)

❑ Direction cosines of $\overrightarrow{OP}$ is cos α, cos β, cos γ where α, β, γ are the angles made by the vector with the positive direction of *x, y,* and *z* axes.

❑ $l = \cos\alpha = \frac{x}{r}, \ m = \cos\beta = \frac{y}{r}, \ n = \cos\gamma = \frac{z}{r}$ where $r = |\vec{r}| = |\overrightarrow{OP}|$

❑ $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$ where *a*, *b*, *c* are the direction ratios.

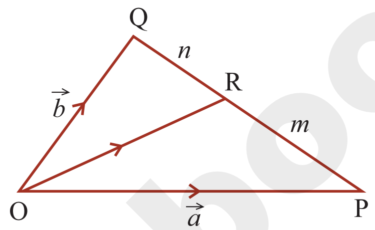
❑ 

$\overline{AB} + \overline{BC} = \overline{AC}$ [triangle law of vector addition]

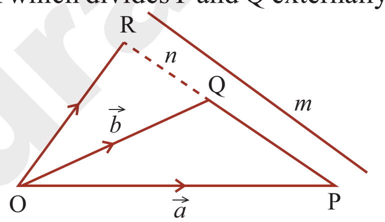
❑ $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$

❑ $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ (or) $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ [Parallelogram law of addition]

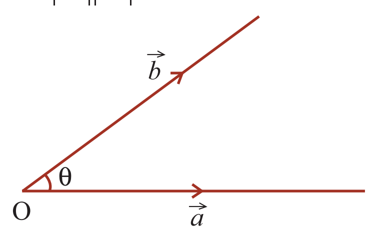
- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (commutative property)
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (Associative property)
- $|\lambda \vec{a}| = |\lambda| |\vec{a}|$ where λ is a scalar.
- Unit vector $\hat{a} = \frac{1}{|\vec{a}|}\vec{a}$.
- If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are any two points, then $\overrightarrow{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- The position vector of the point R which divides $P(\vec{a})$ and $Q(\vec{b})$ internally in the ration $m : n$ is $\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n}$



- The position vector of the point R which divides P and Q externally in the ratio $m : n$ is $\overrightarrow{OR} = \frac{m\vec{b} - n\vec{a}}{m+n}$

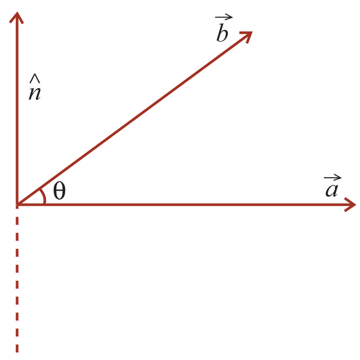


- If R is the mid-point of PQ, then $\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$
- Scalar product of two vectors $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ Where θ is the angle between $\vec{a}$ and $\vec{b}$ $0 \le \theta \le \pi$.



- For mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, we have $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$
- Projection of a vector $\vec{a}$ on the other vector $\vec{b}$ is given by $\vec{a}\cdot\vec{b}$ (OR) $\vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|}$ (OR) $\dfrac{1}{|\vec{b}|}\vec{a}\cdot\vec{b}$
- If α, β, γ, are the direction angles of the vector $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, then its direction cosines are $\cos\alpha = \dfrac{\vec{a}\cdot\hat{c}}{|\vec{a}||\hat{c}|} = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{a_2}{|\vec{a}|}$, $\cos\gamma = \dfrac{a_3}{|\vec{a}|}$.

  Work done $\vec{F}\cdot\vec{d}$ where $\vec{F}$ is the force and $\vec{d}$ is the displacement
- Vector product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$
- $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$ and $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$



- Also $\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{i}\times\hat{k} = -\hat{j}$
- $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$
- $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle then its area is $\dfrac{1}{2}|\vec{a}\times\vec{b}|$
- Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- Torque = $\vec{r}\times\vec{F}$ where $\vec{F}$ is the force.

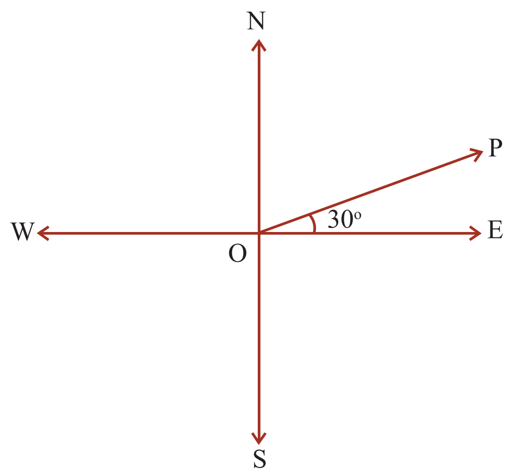
TEXTUAL QUESTIONS

EXERCISE 8.1

1. Represent graphically the displacement of (i) 45cm 30° north of east. (ii) 80km, 60° south of west

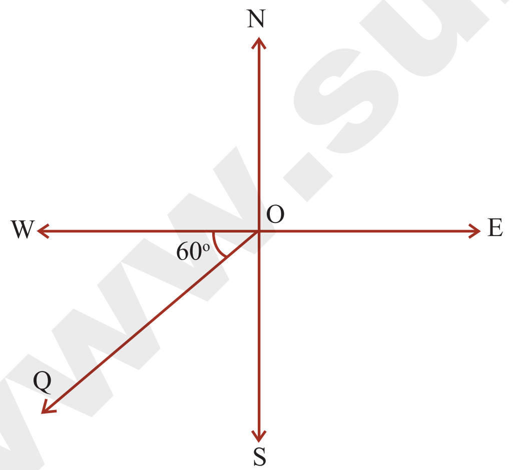
Solution :

(i) 45 cm 30° north of east.



The vector $\overrightarrow{OP}$ represents a displacement of 45 cm, 30° north of east.

(ii) 80 km, 60° south of west.



The vector $\overrightarrow{OQ}$ represents a displacement of 80 km, 60° south of west.

2. Prove that the relation R defined on the set V of all vectors by $\vec{a}$ R $\vec{b}$ if $\vec{a} = \vec{b}$, is an equivalence relation on V.

Solution : Let $\vec{a}, \vec{b}, \vec{c} \in$ V, where V is the set of all vectors.

Let R be the relation defined by $\vec{a} = \vec{b}$

(i) **Reflexive:** $\vec{a} = \vec{a} \Rightarrow aRa \Rightarrow$ R is Reflexive.

(ii) **Symmetric:** $\vec{a} = \vec{b} \Rightarrow \vec{b} = \vec{a}$

$\therefore aRb \Rightarrow bRa \Rightarrow$ R is Symmetric.

(iii) **Transitive:** $\vec{a} = \vec{b}, \vec{b} = \vec{c} \Rightarrow \vec{a} = \vec{c}$

$\therefore aRb, bRc \Rightarrow aRc$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

3. Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B. Prove that the position vectors of the points which trisects the line segment AB are $\dfrac{\vec{a}+2\vec{b}}{3}$ and $\dfrac{\vec{b}+2\vec{a}}{3}$.

Solution :

Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B.

$\Rightarrow \overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$.

Let P divides the line segment AB in the ratio 1:2 and Q divides the line segment AB in the ratio 2:1

$$\therefore \overrightarrow{OP} = \frac{1.(\overrightarrow{OB}) + 2(\overrightarrow{OA})}{1+2}$$

$$= \frac{1(\vec{b}) + 2(\vec{a})}{3} = \frac{\vec{b}+2\vec{a}}{3}$$

$$\text{and } \overrightarrow{OQ} = \frac{2(\overrightarrow{OB}) + 1(\overrightarrow{OA})}{2+1}$$

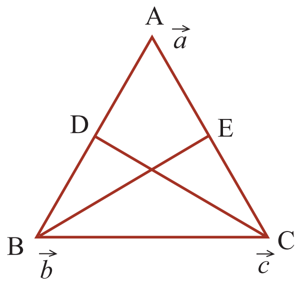
$$= \frac{2\vec{b}+\vec{a}}{3} = \frac{\vec{a}+2\vec{b}}{3}$$

Hence, the required position vectors are $\dfrac{\vec{b}+2\vec{a}}{3}$ and $\dfrac{\vec{a}+2\vec{b}}{3}$.

4. If D and E are the midpoints of the sides AB and AC of a triangle ABC, prove that $\vec{BE}+\vec{DC}=\frac{3}{2}\vec{BC}$.

Solution : Let the position vectors of the vertices of the ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$$\Rightarrow \vec{OA}=\vec{a},\ \vec{OB}=\vec{b} \text{ and } \vec{OC}=\vec{c}$$



Since D is the mid-point of the side AB,

$$\vec{OD}=\frac{\vec{a}+\vec{b}}{2} \quad \ldots (1)$$

and E is the mid-point of the AC

$$\Rightarrow \vec{OE}=\frac{\vec{a}+\vec{c}}{2} \quad \ldots (2)$$

$$\vec{BE}=\vec{OE}-\vec{OB}=\frac{\vec{a}+\vec{c}}{2}-\vec{b}=\frac{\vec{a}+\vec{c}-2\vec{b}}{2}$$

[From (2)]

$$\vec{DC}=\vec{OC}-\vec{OD}=\vec{c}-\frac{\vec{a}+\vec{b}}{2}=\frac{2\vec{c}-\vec{a}-\vec{b}}{2}$$

$$\therefore \vec{BE}+\vec{DC}=\frac{\vec{a}+\vec{c}-2\vec{b}}{2}+\frac{2\vec{c}-\vec{a}-\vec{b}}{2}$$

$$=\frac{\vec{a}+\vec{c}-2\vec{b}+2\vec{c}-\vec{a}-\vec{b}}{2}$$

$$=\frac{3\vec{c}-3\vec{b}}{2}=\frac{3}{2}(\vec{c}-\vec{b})$$

$$=\frac{3}{2}(\vec{OC}-\vec{OB})$$

$$\therefore \vec{BE}+\vec{DC}=\frac{3}{2}\vec{BC}$$

Hence proved.

5. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side whose length is half of the length of the third side.

Solution : Let the position vectors of the vertices of the triangle be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$$\vec{OA}=\vec{a},\ \vec{OB}=\vec{b} \text{ and } \vec{OC}=\vec{c}.$$

Since D is the mid-point of AB,

$$\vec{OD}=\frac{\vec{OA}+\vec{OB}}{2}=\frac{\vec{a}+\vec{b}}{2}$$

Also E is the mid-point of AC,

$$\vec{OE}=\frac{\vec{OA}+\vec{OC}}{2}=\frac{\vec{a}+\vec{c}}{2}$$

$$\vec{DE}=\vec{OE}-\vec{OD}=\frac{\vec{a}+\vec{c}}{2}-\frac{\vec{a}+\vec{b}}{2}$$

$$=\frac{\vec{a}+\vec{c}-\vec{a}-\vec{b}}{2}=\frac{\vec{c}-\vec{b}}{2}$$

$$=\frac{1}{2}(\vec{OC}-\vec{OB})=\frac{1}{2}(\vec{BC})$$

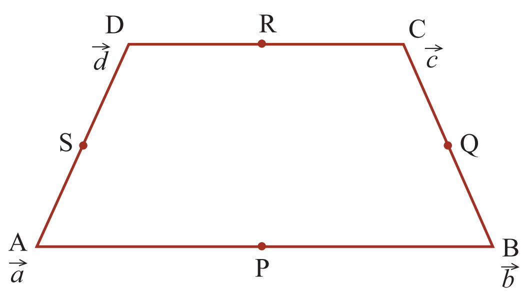
$$\Rightarrow \vec{DE}=\lambda(\vec{BC}) \text{ where } \lambda=\frac{1}{2}$$

$$\therefore \vec{DE}\parallel\vec{BC} \text{ and } \vec{DE}=\frac{1}{2}(\vec{BC})$$

Hence, $\vec{DE}$ is parallel to $\vec{BC}$ and whose length is half of the length of the third side.

6. Prove that the line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

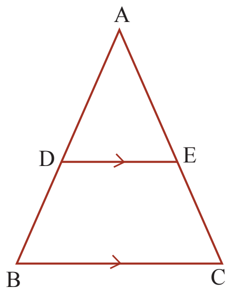
Solution :



Let the position vectors of the vertices of the quadrilateral be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

Let P, Q, R, S be the mid-points of the adjacent sides of the quadrilateral.

To prove that PQRS is a parallelogram.



$$\overrightarrow{OP} = \frac{\vec{a}+\vec{b}}{2}, \quad \overrightarrow{OQ} = \frac{\vec{b}+\vec{c}}{2},$$

$$\overrightarrow{OR} = \frac{\vec{c}+\vec{d}}{2}, \quad \overrightarrow{OS} = \frac{\vec{a}+\vec{d}}{2}$$

Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \frac{\vec{b}+\vec{c}}{2} - \frac{\vec{a}+\vec{b}}{2}$$

$$= \frac{\vec{b}+\vec{c}-\vec{a}-\vec{b}}{2}$$

$$= \frac{\vec{c}-\vec{a}}{2} \quad \ldots (1)$$

$$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS}$$

$$= \frac{\vec{c}+\vec{d}}{2} - \frac{\vec{a}+\vec{d}}{2}$$

$$= \frac{\vec{c}+\vec{d}-\vec{a}-\vec{d}}{2}$$

$$= \frac{\vec{c}-\vec{a}}{2} \quad \ldots (2)$$

From (1) and (2), $\overrightarrow{PQ} = \overrightarrow{SR}$

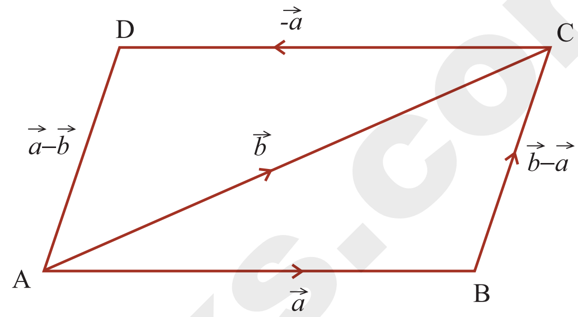
and $\overrightarrow{PQ} = 1(\overrightarrow{SR}) \Rightarrow \overrightarrow{PQ} \parallel \overrightarrow{SR}$

Thus, one pair of parallel sides of PQRS are parallel and equal.

∴ PQRS is a parallelogram.

7. If $\vec{a}$ and $\vec{b}$ represent a side and a diagonal of a parallelogram, find the other sides and the other diagonal.

Solution : Let ABCD be the parallelogram.



Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$.

In ΔABC, $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$\Rightarrow \vec{b} = \vec{a} + \overrightarrow{BC}$

$\Rightarrow \overrightarrow{BC} = \vec{b} - \vec{a}$

Since ABCD is a parallelogram,

$$\overrightarrow{CD} = -\overrightarrow{AB}$$

$$\overrightarrow{CD} = -\vec{a}$$

$$\overrightarrow{DA} = -(\overrightarrow{BC}) = -(\vec{b}-\vec{a})$$

$\Rightarrow \overrightarrow{DA} = \vec{a} - \vec{b}$.

In Δ BCD, $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \vec{b} - \vec{a} - \vec{a} = \vec{b} - 2\vec{a}$

Hence, the other sides of the parallelogram are $\vec{b} - \vec{a}$, $-\vec{a}$, $\vec{a} - \vec{b}$ and the other diagonal is $\vec{b} - 2\vec{a}$.

8. If $\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR}$, prove that the points P, Q, R are collinear.

Solution : Given $\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR}$

$\Rightarrow \overrightarrow{PQ} = \overrightarrow{QR}$

[By triangle law of addition]

$\Rightarrow \overrightarrow{PQ} = \overrightarrow{QR}$

and Q is a common point.

Hence, the points P, Q, R are collinear.

9. If D is the midpoint of the side BC of a triangle ABC, prove that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$.

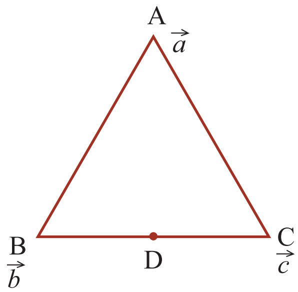
Solution : Let the position vector of the vertices of the triangle be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore \overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

Since D is the mid-point of BC,

$$\overrightarrow{OD} = \frac{\vec{b}+\vec{c}}{2} \quad \ldots (1)$$

To prove that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$



$$\begin{aligned}
\text{LHS} &= \overrightarrow{AB} + \overrightarrow{AC} \\
&= \overrightarrow{OB} - \overrightarrow{OA} + \overrightarrow{OC} - \overrightarrow{OA} \\
&= \vec{b} - \vec{a} + \vec{c} - \vec{a} \\
&= \vec{b} + \vec{c} - 2\vec{a} \\
\text{RHS} &= 2\overrightarrow{AD} \\
&= 2(\overrightarrow{OD} - \overrightarrow{OA}) \\
&= 2\left(\frac{\vec{b}+\vec{c}}{2} - \vec{a}\right) \quad \text{[From (1)]} \\
&= 2\left(\frac{\vec{b}+\vec{c}-2\vec{a}}{2}\right) = \vec{b} + \vec{c} - 2\vec{a} \\
\therefore \text{LHS} &= \text{RHS}
\end{aligned}$$

Hence proved.

10. If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$. **[Hy - 2018]**

Solution : Let the position vector of the vertices of the ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\therefore \overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$.

Since G is the centroid of ΔABC, we have

$$\Rightarrow \overrightarrow{OG} = \frac{\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}}{3}$$

$$\Rightarrow 3\overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} \quad \ldots (1)$$

$$\begin{aligned}
\text{Now, LHS} &= \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} \\
&= \overrightarrow{OA} - \overrightarrow{OG} + \overrightarrow{OB} - \overrightarrow{OG} + \overrightarrow{OC} - \overrightarrow{OG} \\
&= (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) - 3\overrightarrow{OG} \\
&= (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) - (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) = \vec{0} \\
&= \text{RHS}
\end{aligned}$$

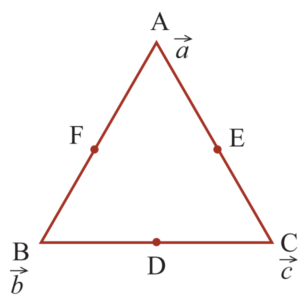
Hence proved.

11. Let A, B, and C be the vertices of a triangle. Let D, E, and F be the midpoints of the sides BC, CA, and AB respectively. Show that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$.

Solution : Let the position vector of the vertices of the ΔABC be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

Since D is the mid-point of BC.

$$\Rightarrow \overrightarrow{OD} = \frac{\vec{b}+\vec{c}}{2}$$



E is the mid-point of AC,

$$\Rightarrow \overrightarrow{OE} = \frac{\vec{a}+\vec{c}}{2}$$ and F is the mid-point of AB

$$\overrightarrow{OF} = \frac{\vec{a}+\vec{b}}{2}$$

To prove that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$

$$\text{LHS} = \overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}$$
$$= \overrightarrow{OD}-\overrightarrow{OA}+\overrightarrow{OE}-\overrightarrow{OB}+\overrightarrow{OF}-\overrightarrow{OC}$$
$$= \frac{\vec{b}+\vec{c}}{2}-\vec{a}+\frac{\vec{a}+\vec{c}}{2}-\vec{b}+\frac{\vec{a}+\vec{b}}{2}-\vec{c}$$
$$= \frac{\vec{b}+\vec{c}-2\vec{a}+\vec{a}+\vec{c}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}}{2}$$
$$= \frac{\vec{0}}{2}=\vec{0}=\text{RHS}$$ Hence proved.

12. If ABCD is a quadrilateral and E and F are the midpoints of AC and BD respectively, then prove that $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=4\overrightarrow{EF}$.

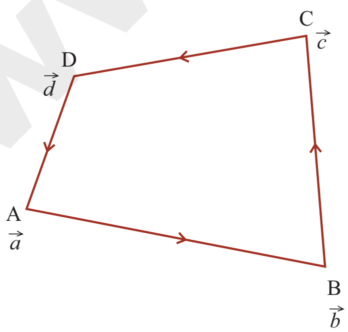
Solution : Let the position vector of the vertices of the quadrilateral ABCD be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

$$\therefore \overrightarrow{OA}=\vec{a},\ \overrightarrow{OB}=\vec{b},$$
$$\overrightarrow{OC}=\vec{c} \text{ and } \overrightarrow{OD}=\vec{d}.$$

Since E and F are the mid-points of AC and BD respectively, we have

$$\overrightarrow{OE}=\frac{\vec{a}+\vec{c}}{2} \text{ and}$$
$$\overrightarrow{OF}=\frac{\vec{b}+\vec{d}}{2} \quad \ldots (1)$$

To prove that $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=4\overrightarrow{EF}$



$$\text{LHS} = \overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}$$
$$= \overrightarrow{OB}-\overrightarrow{OA}+\overrightarrow{OD}-\overrightarrow{OA}+\overrightarrow{OB}-\overrightarrow{OC}+\overrightarrow{OD}-\overrightarrow{OC}$$
$$= \vec{b}-\vec{a}+\vec{d}-\vec{a}+\vec{b}-\vec{c}+\vec{d}-\vec{c}$$
$$= -2\vec{a}+2\vec{b}-2\vec{c}+2\vec{d}$$
$$= 2[(\vec{b}+\vec{d})-(\vec{a}+\vec{c})]$$
$$= 2[2\overrightarrow{OF}-2\overrightarrow{OE}] \text{ [From (1)]}$$
$$= 4.[\overrightarrow{OF}-\overrightarrow{OE}]=4.\overrightarrow{EF}=\text{RHS}$$

Hence proved.

EXERCISE 8.2

1. Verify whether the following ratios are direction cosines of some vector or not.

(i) $\frac{1}{5},\frac{3}{5},\frac{4}{5}$ (ii) $\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}$ (iii) $\frac{4}{3},0,\frac{3}{4}$

Solution :

(i) Given ratios are $\frac{1}{5}$, $\frac{3}{5}$ and $\frac{4}{5}$.

Let the ratios are $l=\frac{1}{5}$, $m=\frac{3}{5}$, $n=\frac{4}{5}$

$$\therefore l^2+m^2+n^2=\left(\frac{1}{5}\right)^2+\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2=\frac{1}{25}+\frac{9}{25}+\frac{16}{25}$$
$$=\frac{26}{25}\neq 1$$

Hence, the given ratios are not the direction cosines of any vector.

(ii) Let $l=\frac{1}{\sqrt{2}}$, $m=\frac{1}{2}$ and $n=\frac{1}{2}$

$$\therefore l^2+m^2+n^2=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2$$
$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{2+1+1}{4}=\frac{4}{4}=1$$

Hence, the given ratios are direction cosines of some vector.

(iii) Let $l=\frac{4}{3}$, $m=0$, $n=\frac{3}{4}$

$$\therefore l^2+m^2+n^2=\left(\frac{4}{3}\right)^2+0^2+\left(\frac{3}{4}\right)^2$$

$= \frac{16}{9}+\frac{9}{16} = \frac{256+81}{16\times 9} = \frac{337}{144} \neq 1$

Hence, the given ratios are not the direction cosines of any vector.

2. Find the direction cosines of a vector whose direction ratios are (i) 1 , 2 , 3 (ii) 3 , – 1 , 3 (iii) 0 , 0 , 7

Solution :

(i) Given direction ratios are 1, 2, 3

Let $x = 1, y = 2, z = 3$

$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+4+9} = \sqrt{14}$

The direction cosines are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$

Thus, the direction cosines are $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

(ii) Let $x = 3, y = -1, z = 3$

$\therefore r = \sqrt{x^2+y^2+z^2} = \sqrt{9+1+9} = \sqrt{19}$

Hence, the direction consines are

$\frac{3}{\sqrt{19}}, \frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}$

(iii) Let $x = 0, y = 0, z = 7$

$\therefore r = \sqrt{x^2+y^2+z^2} = \sqrt{0+0+7^2} = 7$

Hence, the direction consines are $\frac{0}{7}, \frac{0}{7}, \frac{7}{7}$

$\Rightarrow 0, 0, 1.$

3. Find the direction cosines and direction ratios for the following vectors.

(i) $3\hat{i} - 4\hat{j} + 8\hat{k}$ **(ii)** $3\hat{i} + \hat{j} + \hat{k}$

(iii) $\hat{j}$ **(iv)** $5\hat{i} - 3\hat{j} - 48\hat{k}$

(v) $3\hat{i} - 3\hat{k} + 4\hat{j}$ **(vi)** $\hat{i} - \hat{k}$

Solution :

(i) Given vector is $3\hat{i}-4\hat{j}+8\hat{k}$

The direction ratios of $3\hat{i}-4\hat{j}+8\hat{k}$ are $3, -4, 8$.

$r = \sqrt{x^2+y^2+z^2}$

$= \sqrt{3^2+(-4)^2+8^2}$

$= \sqrt{9+16+64} = \sqrt{89}$

Hence, its direction cosines are $\frac{3}{\sqrt{89}}, \frac{-4}{\sqrt{89}}, \frac{8}{\sqrt{89}}$

(ii) Give vector is $3\hat{i}+\hat{j}+\hat{k}$

The direction ratios of $3\hat{i}+\hat{j}+\hat{k}$ are 3, 1, 1.

$r = \sqrt{x^2+y^2+z^2}$

$= \sqrt{3^2+1^2+1^2} = \sqrt{11}$

Hence, its direction cosines are $\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$

(iii) Given vector is $\hat{j}$

The direction ratios of $\hat{j}$ are 0, 1, 0.

$x = \sqrt{x^2+y^2+z^2}$

$= \sqrt{0+1^2+0} = 1$

Hence, its direction cosines are $\frac{0}{1}, \frac{1}{1}, \frac{0}{1} \Rightarrow 0,$ 1, 0.

(iv) The given vector is $5\hat{i}-3\hat{j}-48\hat{k}$

The direction ratios are 5, – 3, – 48.

$r = \sqrt{x^2+y^2+z^2}$

$= \sqrt{5^2+(-3)^2+(-48)^2}$

$= \sqrt{25+9+2304} = \sqrt{2338}$

Hence, the direction cosines are

$\frac{5}{\sqrt{2338}}, \frac{-3}{\sqrt{2338}}, \frac{-48}{\sqrt{2338}}$

(v) The given vector is $3\hat{i}-3\hat{k}+4\hat{j}$

$\Rightarrow 3\hat{i}+4\hat{j}-3\hat{k}$

The direction ratios are 3, 4, – 3.

$r = \sqrt{x^2+y^2+z^2}$

$= \sqrt{3^2+4^2+(-3)^2}$

$= \sqrt{9+16+9} = \sqrt{34}$

Hence, the direction cosines are $\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{-3}{\sqrt{34}}$

(vi) The given vector is $\hat{i}-\hat{k}$

The direction ratios are 1, 0, – 1.

$x = \sqrt{x^2+y^2+z^2}$

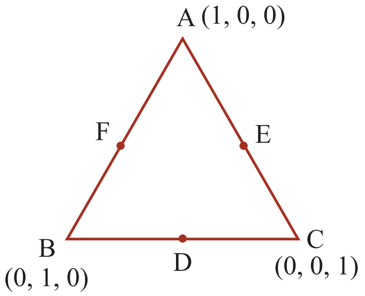
$= \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$

Hence, the direction cosines are $\frac{1}{\sqrt{2}}, \frac{0}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

$\Rightarrow \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$

4. A triangle is formed by joining the points (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find the direction cosines of the medians.

Solution : Let the vertices of the triangle be A(1, 0, 0), B(0, 1, 0), C(0, 0, 1).



Let D, E, F are the mid-point of the sides BC, CA and AB respectively.

$$\therefore \text{ D is } \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$\Rightarrow \text{ D is } \left(0, \frac{1}{2}, \frac{1}{2}\right) \text{ and E is } \left(\frac{1}{2}, 0, \frac{1}{2}\right),$$

$$\text{F is } \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$\text{Medians } \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$

$$= \left(0\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}\right) - \left(\hat{i} - 0\hat{j} + 0\hat{k}\right)$$

$$= -\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+\frac{1}{4}+\frac{1}{4}}$$

$$= \sqrt{\frac{4+1+1}{4}} = \frac{\sqrt{6}}{2}$$

Hence, the direction cosines of $\overrightarrow{AD}$ are,

$$\frac{-1}{\frac{\sqrt{6}}{2}}, \frac{1}{\cancel{2}\times\frac{\sqrt{6}}{\cancel{2}}}, \frac{1}{\cancel{2}\times\frac{\sqrt{6}}{\cancel{2}}} \Rightarrow \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

$$\text{The median } \overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB}$$

$$= \left(\frac{1}{2}\hat{i} - 0\hat{j} + \frac{1}{2}\hat{k}\right) - \left(0\hat{i} + \hat{j} + 0\hat{k}\right)$$

$$= \frac{1}{2}\hat{i} - \hat{j} + \frac{1}{2}\hat{k}$$

$$r = \sqrt{\frac{1}{4}+1+\frac{1}{4}} = \sqrt{\frac{1+4+1}{4}} = \frac{\sqrt{6}}{2}$$

The direction cosines of $\overrightarrow{BE}$ are $\frac{1}{\cancel{2}\times\frac{\sqrt{6}}{\cancel{2}}}, -\frac{1}{\frac{\sqrt{6}}{2}}, \frac{1}{\cancel{2}\times\frac{\sqrt{6}}{\cancel{2}}}$

$$\Rightarrow \text{The median } \overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC}$$

$$= \left(\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} + 0\hat{k}\right) - \left(0\hat{i} + 0\hat{j} + \hat{k}\right)$$

$$= \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j} - \hat{k}$$

$$r = \sqrt{\frac{1}{4}+\frac{1}{4}+1} = \frac{\sqrt{6}}{2}$$

$\therefore$ The direction of cosines of $\overrightarrow{CF}$ are $\frac{1}{\cancel{2}\times\frac{\sqrt{6}}{\cancel{2}}}, \frac{1}{\cancel{2}\times\frac{\sqrt{6}}{\cancel{2}}}, \frac{-1}{\frac{\sqrt{6}}{\cancel{2}}}$

$$\Rightarrow \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$$

5. If $\frac{1}{2}, \frac{1}{\sqrt{2}}$, *a* are the direction cosines of some vector, then find *a*.

Solution : Given direction cosines of some vector are $\frac{1}{2}, \frac{1}{\sqrt{2}}, a$

$$\text{Let } l = \frac{1}{2}, m = \frac{1}{\sqrt{2}}, n = a$$

$$\text{We know that } l^2+m^2+n^2 = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + a^2 = 1$$

$$\Rightarrow \frac{1}{4}+\frac{1}{2}+a^2 = 1$$

$$\Rightarrow a^2 = 1-\frac{1}{4}-\frac{1}{2}$$

$$= \frac{4-1-2}{4} = \frac{1}{4}$$

$$a = \pm\sqrt{\frac{1}{4}}$$

$$\Rightarrow a = \pm\frac{1}{2}$$

6. If (*a*, *a* + *b* , *a* + *b* + *c*) is one set of direction ratios of the line joining (1, 0, 0) and (0, 1, 0), then find a set of values of *a*, *b*, *c*.

Solution : Given points are A(1, 0, 0) and B(0, 1,0) and one set of direction ratios are a, $a+b$, $a+b+c$.

Case (i):

$$\vec{AB} = \vec{OB} - \vec{OA} = (0\hat{i}+\hat{j}+0\hat{k})-(\hat{i}+0\hat{j}+0\hat{k}) = -\hat{i}+\hat{j}$$

$\therefore$ Direction ratios of the line $\vec{AB}$ are (–1, 1, 0)

Given (–1, 1, 0) $= (a, a+b, a+b+c)$

Equating the like components both sides, we get

$$a = -1,\ a+b = 1,\ a+b+c = 0$$
$$a = -1,\ -1+b = 1 \Rightarrow b = 2$$
$$-1+2+c = 0 \Rightarrow c = -1$$
$$\therefore a = -1, b = 2, c = -1$$

Case (ii):

$$\vec{BA} = \vec{OA} - \vec{OB} = (\hat{i}+0\hat{j}+0\hat{k})-(0\hat{i}+\hat{j}+0\hat{k}) = \hat{i}-\hat{j}$$

$\therefore$ Direction ratios of the line $\vec{BA}$ are (1, –1, 0)

Given (1, –1, 0) $= (a, a+b, a+b+c)$

Equating the like components both sides, we get

$$a = 1,\ a+b = -1,\ a+b+c = 0$$
$$a = 1,\ 1+b = -1 \Rightarrow b = -2$$
$$1-2+c = 0 \Rightarrow c = 1$$
$$\therefore a = 1, b = -2, c = 1$$

7. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}-4\hat{j}-4\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ form a right angled triangle.

Solution : Let the sides of the triangle be

$$\hat{a} = 2\hat{i}-\hat{j}+\hat{k},\ \vec{b} = 3\hat{i}-4\hat{j}-4\hat{k},$$
$$\hat{c} = \hat{i}-3\hat{j}-5\hat{k}$$
$$|\hat{a}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$$
$$|\vec{b}| = \sqrt{3^2+(-4)^2+(-4)^2} = \sqrt{9+16+16} = \sqrt{41}$$
$$|\hat{c}| = \sqrt{1^2+(-3)^2+(-5)^2} = \sqrt{1+9+25} = \sqrt{35}$$

Now $|\vec{b}|^2 = \left(\sqrt{41}\right)^2 = 41 = 35+6$

$$= \left(\sqrt{35}\right)^2 + \left(\sqrt{6}\right)^2 = |\hat{a}|^2 + |\hat{c}|^2$$

By Pythagoras theorem, the given vectors form a right angled triangle.

8. Find the value of λ for which the vectors $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}+\lambda\hat{j}+3\hat{k}$ are parallel.

Solution : Given $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$, $\vec{b} = \hat{i}+\lambda\hat{j}+3\hat{k}$

Given $\vec{a} \parallel \vec{b}$

$$\therefore \vec{a} = (\text{some scalar})\,\vec{b}$$
$$\Rightarrow \vec{a} = 3\hat{i}+2\hat{j}+9\hat{k} = 3\left(\hat{i}+\frac{2}{3}\hat{j}+3\hat{k}\right)$$
$$\vec{a} = 3(\vec{b})$$
$$\vec{b} = \hat{i}+\frac{2}{3}\hat{j}+3\hat{k}$$

Comparing this with $\hat{i}+\lambda\hat{j}+3\hat{k}$ we get

$$\lambda = \frac{2}{3}$$

9. Show that the following vectors are coplanar

(i) $\hat{i}-2\hat{j}+3\hat{k}$, $-2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{j}+2\hat{k}$

(ii) $2\hat{i}+3\hat{i}+\hat{k}$, $\hat{i}-\hat{j}$, $7\hat{i}+3\hat{j}+2\hat{k}$. **[Hy - 2018]**

Solution : Let $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$, $\vec{b} = -2\hat{i}+3\hat{j}-4\hat{k}$,

$$\vec{c} = -\hat{j}+2\hat{k}$$

Let $\vec{a} = s\vec{b}+t\vec{c}$

$$\Rightarrow \hat{i}-2\hat{j}+3\hat{k} = s(-2\hat{i}+3\hat{j}-4\hat{k})+t(-\hat{j}+2\hat{k})$$
$$\Rightarrow \hat{i}-2\hat{j}+3\hat{k} = (-2s)\hat{i}+(3s-t)\hat{j}+(-4s+2t)\hat{k}$$

Equating the like components both sides, we get

$$-2s = 1 \quad \ldots (1)$$
$$3s-t = -2 \quad \ldots (2)$$
$$-4s+2t = 3 \quad \ldots (3)$$

From (1), $s = -\frac{1}{2}$

Substituting $s = -\frac{1}{2}$ in (2) we get,

$$3\left(\frac{-1}{2}\right)-t = -2 \Rightarrow -\frac{3}{2}-t = -2$$

$$-t = -2+\frac{3}{2}$$

$$-t = \frac{-4+3}{2} = \frac{-1}{2}$$

$$t = \frac{1}{2}$$

Substituting $s = -\frac{1}{2}, t = \frac{1}{2}$ in (3) we get,

$$-4\left(\frac{-1}{2}\right)+2\left(\frac{1}{2}\right) = 3$$

$$\Rightarrow \quad 2+1 = 3$$

$$\Rightarrow \quad 3 = 3$$

which satisfies equation (3).

Thus, one vector is a linear combination of other two vectors.

Hence, the given vectors are co-planar.

(ii) Let $\vec{a} = -2\hat{i}+3\hat{j}+\hat{k}$

$$\vec{b} = \vec{i}-\vec{j}$$

$$\vec{c} = 7\hat{i}+3\hat{j}+2\hat{k}$$

Let $\vec{a} = s\vec{b}+t\vec{c}$

where s and t are scalars

$$\Rightarrow \quad 2\hat{i}+3\hat{j}+\hat{k} = s(\vec{i}-\vec{j})+t(7\hat{i}+3\hat{j}+2\hat{k})$$

$$\Rightarrow \quad 2\hat{i}+3\hat{j}+\hat{k} = \hat{i}\,(s+7t)+\hat{j}\,(-s+3t)+\hat{k}\,(2t)$$

Equating the like components both sides, we get

$$2 = s+7t \quad \text{... (1)}$$

$$3 = -s+3t \quad \text{... (2)}$$

$$1 = 2t \quad \text{... (3)}$$

Let us solve (2) (3), to get the values of s and t.

From (3), $t = \frac{1}{2}$

Substituting $t = \frac{1}{2}$ in (2) we get,

$$3 = -s+3\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad 3 = -s+\frac{3}{2}$$

$$\Rightarrow \quad s = \frac{3}{2}-3 = \frac{3-6}{2} = \frac{-3}{2}$$

$$\therefore t = \frac{1}{2}, s = -\frac{3}{2}$$

Substituting $s = -\frac{3}{2}$ and $t = \frac{1}{2}$ in (1) we get,

$$\Rightarrow \quad 2 = -\frac{3}{2}+7\left(\frac{1}{2}\right) \Rightarrow 2 = -\frac{3}{2}+\frac{7}{2}$$

$$\Rightarrow \quad 2 = \frac{-3+7}{2} \Rightarrow 2 = \frac{4}{2} \Rightarrow 2 = 2$$

The value of s and t satisfy equation (1)

One vector is a linear combination of other two vectors.

Hence, the given vectors are co-planar.

10. Show that the points whose position vectors $4\hat{i}+5\hat{j}+\hat{k}$, $-\hat{j}-\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $-4\hat{i}+4\hat{j}+4\hat{k}$ are coplanar.

Solution : Let the position vectors of the given vector be

$$\overrightarrow{OA} = 4\hat{i}+5\hat{j}+\hat{k}$$

$$\overrightarrow{OB} = -\hat{j}-\hat{k}$$

$$\overrightarrow{OC} = 3\hat{i}+9\hat{j}+4\hat{k} \text{ and}$$

$$\overrightarrow{OD} = -4\hat{i}+4\hat{j}+4\hat{k}$$

Let $\vec{a} = \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$

$$= (-\hat{j}-\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -4\hat{i}-6\hat{j}-2\hat{k}$$

$$\vec{b} = \overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA}$$

$$= (3\hat{i}+9\hat{j}+4\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -\hat{i}+4\hat{j}+3\hat{k} \text{ and}$$

$$\vec{c} = \overrightarrow{AD} = \overrightarrow{OD}-\overrightarrow{OA}$$

$$= (-4\hat{i}+4\hat{j}+4\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -8\hat{i}-\hat{j}+3\hat{k}$$

Also, let $\vec{a} = s\vec{b}+t\vec{c} = -4\hat{i}-6\hat{j}-2\hat{k}$

$$= s(-\hat{i}+4\hat{j}+3\hat{k})+t(-8\hat{i}-\hat{j}+3\hat{k})$$

$$-4\hat{i}-6\hat{j}-2\hat{k} = (-s-8t)\,\hat{i}+(4s-t)\,\hat{j}+(3s+3t)\,\hat{k}$$

Equating the like components on both sides, we get

$$-4 = -s-8t \quad \text{... (1)}$$

$$-6 = 4s-t \quad \text{... (2)}$$

$$-2 = 3s+3t \quad \text{... (3)}$$

(1) × 4 ⇒ $-16 = -4s - 32t$

(2) ⇒ $-6 = 4s - t$

Adding, $-22 = -33t \Rightarrow t = \frac{-22}{-33} = \frac{2}{3}$

Substituting $t = \frac{2}{3}$ in (1) we get,

$-4 = -s - 8\left(\frac{2}{3}\right) \Rightarrow -4 = -s - \frac{16}{3}$

$\Rightarrow s = 4 - \frac{16}{3} = \frac{12-16}{3} = -\frac{4}{3}$

Substituting $t = \frac{2}{3}$, and $s = -\frac{4}{3}$ in (3) we get,

$-2 = 3\left(-\frac{4}{3}\right) + 3\left(\frac{2}{3}\right)$

$\Rightarrow -2 = -4 + 2 \Rightarrow -2 = -2$

which satisfies equation (3).

Thus, one vector is the linear combination of other two vectors.

Hence, the given points are co-planar.

11. If $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$, $\vec{b} = 3\hat{i} - 4\hat{j} - 5\hat{k}$, and $\vec{c} = -3\hat{i} + 2\hat{j} + 3\hat{k}$, find the magnitude and direction cosines of

(i) $\vec{a} + \vec{b} + \vec{c}$ (ii) $3\vec{a} - 2\vec{b} + 5\vec{c}$

Solution : Given $\vec{a} = 2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{b} = 3\hat{i} - 4\hat{j} - 5\hat{k}$ and

$\vec{c} = -3\hat{i} + 2\hat{j} + 3\hat{k}$

(i) $\vec{a} + \vec{b} + \vec{c} = (2\hat{i} + 3\hat{j} - 4\hat{k}) + (3\hat{i} - 4\hat{j} - 5\hat{k}) + (-3\hat{i} + 2\hat{j} + 3\hat{k})$

$= 2\hat{i} + \hat{j} - 6\hat{k}$

$|\vec{a} + \vec{b} + \vec{c}| = \sqrt{2^2 + 1^2 + (-6)^2}$

$= \sqrt{4+1+36} = \sqrt{41}$

Direction cosines of $(\vec{a} + \vec{b} + \vec{c})$ is $\frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{-6}{\sqrt{41}}$

(ii) $3\vec{a} - 2\vec{b} + 5\vec{c} = 3(2\hat{i} + 3\hat{j} - 4\hat{k}) - 2(3\hat{i} - 4\hat{j} - 5\hat{k}) + 5(-3\hat{i} + 2\hat{j} + 3\hat{k})$

$= 6\hat{i} + 9\hat{j} - 12\hat{k} - 6\hat{i} + 8\hat{j} + 10\hat{k} - 15\hat{i} + 10\hat{j} + 15\hat{k}$

$= -15\hat{i} + 27\hat{j} + 13\hat{k}$

$|3\vec{a} - 2\vec{b} + 5\vec{c}| = \sqrt{(-15)^2 + 27^2 + 13^2}$

$= \sqrt{225 + 729 + 169} = \sqrt{1123}$

Direction cosines of $3\vec{a} - 2\vec{b} + 5\vec{c}$ is $\frac{-15}{\sqrt{1123}}, \frac{27}{\sqrt{1123}}, \frac{13}{\sqrt{1123}}$

12. The position vectors of the vertices of a triangle are $\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} - 4\hat{j} + 5\hat{k}$ and $-2\hat{i} + 3\hat{j} - 7\hat{k}$. Find the perimeter of the triangle.

Solution : Let the vertices of the triangle be A, B, C.

Then, given $\vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{OB} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ and

$\vec{OC} = -2\hat{i} + 3\hat{j} - 7\hat{k}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (3\hat{i} - 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 2\hat{i} - 6\hat{j} + 2\hat{k}$

$|\vec{AB}| = \sqrt{2^2 + (-6)^2 + 2^2}$

$= \sqrt{4+36+4} = \sqrt{44}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (-2\hat{i} + 3\hat{j} - 7\hat{k}) - (3\hat{i} - 4\hat{j} + 5\hat{k}) = -5\hat{i} + 7\hat{j} - 12\hat{k}$

$|\vec{BC}| = \sqrt{(-5)^2 + 7^2 + (-12)^2}$

$= \sqrt{25+49+144}$

$= \sqrt{218}$

$\vec{CA} = \vec{OA} - \vec{OC}$

$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (-2\hat{i} + 3\hat{j} - 7\hat{k}) = 3\hat{i} - \hat{j} + 10\hat{k}$

$|\vec{CA}| = \sqrt{3^2 + (-1)^2 + 10^2}$

$= \sqrt{9+1+100} = \sqrt{110}$

∴ Perimeter of ΔABC,

$|\vec{AB}| + |\vec{BC}| + |\vec{CA}| = \left(\sqrt{44} + \sqrt{218} + \sqrt{110}\right)$ units

13. Find the unit vector parallel to $3\vec{a} - 2\vec{b} + 4\vec{c}$, if $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$, and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$.

Solution : Given $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$

$\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$

and $\vec{c} = \hat{i}+2\hat{j}-\hat{k}$

$3\vec{a}-2\vec{b}+4\vec{c} = 3(3\hat{i}-\hat{j}-4\hat{k})-2(-2\hat{i}+4\hat{j}-3\hat{k})+4(\hat{i}+2\hat{j}-\hat{k})$

$= 9\hat{i}-3\hat{j}-12\hat{k}+4\hat{i}-\cancel{8}\hat{j}+6\hat{k}+4\hat{i}+\cancel{8}\hat{j}-4\hat{k}$

$= 17\hat{i}-3\hat{j}-10\hat{k}$

$|3\vec{a}-2\vec{b}+4\vec{c}| = \sqrt{17^2+(-3)^2+(-10)^2}$

$= \sqrt{289+9+100} = \sqrt{398}$

$\therefore$ Unit vector parallel to $(3\vec{a}-2\vec{b}+4\vec{c})$ is

$\frac{1}{\sqrt{398}}(17\hat{i}-3\hat{j}-10\hat{k})$

14. The position vectors $\vec{a}, \vec{b}, \vec{c}$ of three points satisfy the relation $2\vec{a}-7\vec{b}+5\vec{c}=\vec{0}$. Are these points collinear?

Solution : Let the position vector of three points be $\vec{a}, \vec{b}, \vec{c}$.

The given relation is

$2\vec{a}-7\vec{b}+5\vec{c} = \vec{0}$

$\Rightarrow 2\vec{a}+5\vec{c} = 7\vec{b}$

$\Rightarrow \frac{2}{7}\vec{a}+\frac{5}{7}\vec{c} = \vec{b}$

$\Rightarrow s\vec{a}+t\vec{c} = \vec{b}$

Thus, $\vec{b}$ is a linear combination of $\vec{a}$ and $\vec{c}$.

$\therefore$ The given points are collinear.

15. The position vectors of the points P, Q, R, S are $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$, and $\hat{i}-6\hat{j}-\hat{k}$ respectively. Prove that the line PQ and RS are parallel.

Solution : Given $\overrightarrow{OP} = \hat{i}+\hat{j}+\hat{k}$

$\overrightarrow{OQ} = 2\hat{i}+5\hat{j}$

$\overrightarrow{OR} = 3\hat{i}+2\hat{j}-3\hat{k}$

and $\overrightarrow{OS} = \hat{i}-6\hat{j}-\hat{k}$

$\overrightarrow{PQ} = \overrightarrow{OQ}-\overrightarrow{OP}$

$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})$

$= \hat{i}+4\hat{j}-\hat{k}$

$\overrightarrow{RS} = \overrightarrow{OS}-\overrightarrow{OR}$

$= (\hat{i}-6\hat{j}-\hat{k})-(3\hat{i}+2\hat{j}-3\hat{k})$

$= -2\hat{i}-8\hat{j}+2\hat{k}$

$= -2(\hat{i}+4\hat{j}-\hat{k}) = -2\overrightarrow{PQ}$

$\therefore \overrightarrow{RQ} = \lambda\overrightarrow{PQ}$ where $\lambda = -2$

$\therefore \overrightarrow{RQ} \parallel \overrightarrow{PQ}$

16. Find the value or values of *m* for which $m(\hat{i}+\hat{j}+\hat{k})$ is a unit vector.

Solution : Let $\vec{a} = m(\hat{i}+\hat{j}+\hat{k})$

$|\vec{a}| = m\sqrt{1^2+1^2+1^2} = m\sqrt{3}$

To make $\vec{a}$ as a unit vector, $|\vec{a}| = \pm 1$

$\therefore m\sqrt{3} = \pm 1 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$.

17. Show that the points A (1, 1, 1), B(1, 2, 3) and C(2, –1, 1) are vertices of an isosceles triangle.

Solution : Let the position vector of the points A, B, C be

$\overrightarrow{OA} = \hat{i}+\hat{j}+\hat{k}$

$\overrightarrow{OB} = \hat{i}+2\hat{j}+3\hat{k}$

$\overrightarrow{OC} = 2\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}$

$= (\hat{i}+2\hat{j}+3\hat{k})-(\hat{i}+\hat{j}+\hat{k})$

$= \hat{j}+2\hat{k}$

$|\overrightarrow{AB}| = \sqrt{1^2+2^2} = \sqrt{5}$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= (2\hat{i} - \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= \hat{i} - 3\hat{j} - 2\hat{k}$$
$$\therefore |\overrightarrow{BC}| = \sqrt{1^2 + (-3)^2 + (-2)^2}$$
$$= \sqrt{1+9+4} = \sqrt{14}$$
$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$$
$$= (\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$
$$= -\hat{i} + 2\hat{j}$$
$$\therefore |\overrightarrow{CA}| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

Since $|\overrightarrow{AB}| = |\overrightarrow{CA}|$, the given points form an isosceles triangle.

EXERCISE 8.3

1. Find $\vec{a} \cdot \vec{b}$ when

(i) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$

(ii) $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$

Solution :

(i) Given $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$$\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$$
$$\vec{a} \cdot \vec{b} = (\hat{i} - 2\hat{j} + \hat{k}) \cdot (3\hat{i} - 4\hat{j} - 2\hat{k})$$
$$= 1(3) - 2(-4) + 1(-2) = 3 + 8 - 2 = 9$$
$$\therefore \vec{a} \cdot \vec{b} = 9$$

(ii) Given $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$

$$\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$$
$$\vec{a} \cdot \vec{b} = (2\hat{i} + 2\hat{j} - \hat{k}) \cdot (6\hat{i} - 3\hat{j} + 2\hat{k})$$
$$= 12 - 6 - 2 = 12 - 8 = 4$$

2. Find the value λ for which the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, where

(i) $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

(ii) $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$.

Solution :

(i) Given $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$

and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

Since the vectors are perpendicular, $\vec{a} \cdot \vec{b} = 0$

$$\Rightarrow (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$$
$$\Rightarrow 2(1) + \lambda(-2) + 1(3) = 0$$
$$\Rightarrow 2 - 2\lambda + 3 = 0$$
$$\Rightarrow 5 - 2\lambda = 0 \Rightarrow 2\lambda = 5$$
$$\Rightarrow \lambda = \frac{5}{2}$$

(ii) Given $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$

Since the vectors are perpendicular, $\vec{a} \cdot \vec{b} = 0$

$$\Rightarrow (2\hat{i} + 4\hat{j} - \hat{k}) \cdot (3\hat{i} - 2\hat{j} + \lambda\hat{k}) = 0$$
$$\Rightarrow 2(3) + 4(-2) - 1(\lambda) = 0$$
$$\Rightarrow 6 - 8 - \lambda = 0$$
$$\Rightarrow -2 - \lambda = 0$$
$$\Rightarrow \lambda = -2$$

3. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 10, |\vec{b}| = 15$ and $\vec{a} \cdot \vec{b} = 75\sqrt{2}$, find the angle between $\vec{a}$ and $\vec{b}$.

Solution : Given $|\vec{a}| = 10$, $|\vec{b}| = 15$ and $\vec{a} \cdot \vec{b} = 75\sqrt{2}$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

$$\therefore \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{75\sqrt{2}}{10\,(15)} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$= \cos\frac{\pi}{4}.$$
$$\theta = \frac{\pi}{4}.$$

4. Find the angle between the vectors

(i) $2\hat{i} + 3\hat{j} - 6\hat{k}$ and $6\hat{i} - 3\hat{j} + 2\hat{k}$.

(ii) $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$.

Solution :

(i) Let $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$

Let θ be the angle between the given vectors.

$$\vec{a}.\vec{b} = (2\hat{i}+3\hat{j}-6\hat{k}).(6\hat{i}-3\hat{j}+2\hat{k})$$
$$= 12 - 9 - 12 = -9$$
$$|\vec{a}| = \sqrt{2^2+3^2+(-6)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$
and $$|\vec{b}| = \sqrt{6^2+(-3)^2+2^2} = \sqrt{36+9+4} = \sqrt{49} = 7$$
$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-9}{7(7)} = \frac{-9}{49}$$
$$\Rightarrow \theta = \cos^{-1}\left(\frac{-9}{49}\right)$$

(ii) Let $\vec{a} = \hat{i}-\hat{j}$ and $\vec{b} = \hat{j}-\hat{k}$

$$\vec{a}.\vec{b} = (\hat{i}-\hat{j}).(\hat{j}-\hat{k}) = 1(0)-1(1)+0(-1) = -1$$
$$|\vec{a}| = \sqrt{1^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$$
$$|\vec{b}| = \sqrt{1^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$
$$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2} \Rightarrow \cos\theta = -\cos\left(\frac{\pi}{3}\right)$$
$$\Rightarrow \cos\theta = \cos\left(\pi-\frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$$
$$\Rightarrow \theta = \frac{2\pi}{3}$$

5. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+2\vec{b}+\vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$. **[March - 2019]**

Solution : Given $\vec{a}+2\vec{b}+\vec{c} = \vec{0}$

and $|\vec{a}| = 3$, $|\vec{b}| = 4$

and $|\vec{c}| = 7$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

$$\Rightarrow \vec{a}+2\vec{b} = -\vec{c}$$
$$\Rightarrow |\vec{a}+2\vec{b}|^2 = |-\vec{c}|^2$$
$$\Rightarrow |\vec{a}|^2+4|\vec{b}|^2+4(\vec{a}.\vec{b}) = |\vec{c}|^2$$
$$\Rightarrow 9+4(16)+4(\vec{a}.\vec{b}) = 49$$
$$9+64+4(\vec{a}.\vec{b}) = 49$$
$$\Rightarrow 73+4(\vec{a}.\vec{b}) = 49$$
$$\Rightarrow 4(\vec{a}.\vec{b}) = 49-73$$
$$\Rightarrow 4|\vec{a}||\vec{b}|\cos\theta = -24$$
$$\Rightarrow 4(3)(4)\cos\theta = -24$$
$$\Rightarrow \cos\theta = \frac{-24}{4(3)(4)}$$
$$\Rightarrow \cos\theta = \frac{-1}{2} = -\cos\frac{\pi}{3}$$
$$\Rightarrow \cos\theta = \cos\left(\pi-\frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$
$$\Rightarrow \theta = \frac{2\pi}{3}$$

6. Show that the vectors $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$, $\vec{b} = 6\hat{i}+2\hat{j}-3\hat{k}$, and $\vec{c} = 3\hat{i}-6\hat{j}+2\hat{k}$, are mutually orthogonal.

Solution : Given $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$,

$$\vec{b} = 6\hat{i}+2\hat{j}-3\hat{k},$$
$$\vec{c} = 3\hat{i}-6\hat{j}+2\hat{k}$$
$$\vec{a}.\vec{b} = (2\hat{i}+3\hat{j}+6\hat{k}).(6\hat{i}+2\hat{j}-3\hat{k}) = 2(6)+3(2)+6(-3) = 12+6-18 = 0$$
$$\vec{b}.\vec{c} = (6\hat{i}+2\hat{j}-3\hat{k}).(3\hat{i}-6\hat{j}+2\hat{k}) = 6(3)+2(-6)-3(2) = 18-12-6 = 0$$
$$\vec{c}.\vec{a} = (3\hat{i}-6\hat{j}+2\hat{k}).(2\hat{i}+3\hat{j}+6\hat{k}) = 3(2)-6(3)+2(6) = 6-18+12 = 0$$

Since $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ the given vectors are mutually orthogonal.

7. Show that the vectors $-\hat{i}-2\hat{j}-6\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $-\hat{i}+3\hat{j}+5\hat{k}$, form a right angled triangle.

Solution : Let the given vectors are

$$\text{Given } \vec{a} = -\hat{i}-2\hat{j}-6\hat{k},\ \vec{b} = 2\hat{i}-\hat{j}+\hat{k}$$

$$\text{and } \vec{c} = -\hat{i}+3\hat{j}+5\hat{k}$$

$$|\vec{a}| = \sqrt{(-1)^2+(-2)^2+(-6)^2} = \sqrt{1+4+36} = \sqrt{41}$$

$$|\vec{b}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$\text{and } |\vec{c}| = \sqrt{(-1)^2+3^2+5^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$|\vec{a}|^2 = \left(\sqrt{41}\right)^2 = 41 = 35+6 = \left(\sqrt{35}\right)^2+\left(\sqrt{6}\right)^2 = |\vec{b}|^2+|\vec{c}|^2$$

Hence, by Pythagoras theorem, the given vectors form a right angled triangle.

8. If $|\vec{a}|=5, |\vec{b}|=6, |\vec{c}|=7$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.

Solution : Given $|\vec{a}| = 5, |\vec{b}| = 6,$

$$|\vec{c}| = 7 \text{ and } \vec{a}+\vec{b}+\vec{c} = \vec{0}$$

$$|\vec{a}+\vec{b}+\vec{c}| = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b})+2(\vec{b}.\vec{c})+2(\vec{c}.\vec{a})$$

$$= 25+36+49+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = \vec{0}$$

$$\Rightarrow -110 = 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$\Rightarrow \frac{-110}{2} = \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$$

$$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = -55$$

9. Show that the points (2, – 1, 3), (4, 3, 1) and (3, 1, 2) are collinear.

Solution : Let the given points be A(2, –1, 3), B(4, 3, 1) and C(3, 1, 2).

$$\text{Then } \overrightarrow{OA} = 2\hat{i}-\hat{j}+3\hat{k},$$

$$\overrightarrow{OB} = 4\hat{i}+3\hat{j}+\hat{k} \text{ and}$$

$$\overrightarrow{OC} = 3\hat{i}+\hat{j}+2\hat{k}$$

$$\text{Now, } \overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (4\hat{i}+3\hat{j}+\hat{k})-(2\hat{i}-\hat{j}+3\hat{k}) = 2\hat{i}+4\hat{j}-2\hat{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB} = (3\hat{i}+\hat{j}+2\hat{k})-(4\hat{i}+3\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}+\hat{k}$$

$$\overrightarrow{CA} = \overrightarrow{OA}-\overrightarrow{OC} = (2\hat{i}-\hat{j}+3\hat{k})-(3\hat{i}+\hat{j}+2\hat{k}) = -\hat{i}-2\hat{j}+\hat{k}$$

$$\text{Now, } \overrightarrow{AB} = 2\hat{i}+4\hat{j}-2\hat{k} = -2(-\hat{i}-2\hat{j}+\hat{k}) = -2\overrightarrow{BC}$$

Thus $\overrightarrow{AB} \parallel \overrightarrow{BC}$ and B is a common points.

Hence, the given points are collinear.

10. If $\vec{a}, \vec{b}$ are unit vectors and θ is the angle between them, show that

(i) $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}-\vec{b}|$ (ii) $\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

(iii) $\tan\frac{\theta}{2} = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$

Solution : Let $\vec{a}$ and $\vec{b}$ be the unit vectors and θ is the angle between $\vec{a}$ and $\vec{b}$.

(i) Consider $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}.\vec{b})$

$$[\because |\vec{a}| = 1; |\vec{b}| = 1]$$

$$= 1+1-2|\vec{a}||\vec{b}|\cos\theta$$

$$= 2-2\cos\theta$$

$$= 2(1-\cos\theta)$$

$$= 2.2\sin^2\frac{\theta}{2} = 4\sin^2\frac{\theta}{2}$$

$$|\vec{a}-\vec{b}| = 2\sin\frac{\theta}{2}$$

$$\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}-\vec{b}|$$

(ii) Consider $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}.\vec{b})$

$= 1+1+2|\vec{a}||\vec{b}|$

$\cos\theta = 2+2\cos\theta$

$[\because |\vec{a}|=1; |\vec{b}|=1]$

$= 2(1+\cos\theta) = 2.2\cos^2\frac{\theta}{2}$

$= 4\cos^2\frac{\theta}{2}$

$\therefore |\vec{a}+\vec{b}| = 2\cos\frac{\theta}{2}$

$\Rightarrow \cos\frac{\theta}{2} = \frac{1}{2}|\vec{a}+\vec{b}|$

(iii) $\tan\frac{\theta}{2} = \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \dfrac{\frac{1}{2}|\vec{a}-\vec{b}|}{\frac{1}{2}|\vec{a}+\vec{b}|}$ [From (i) and (ii)]

$\tan\frac{\theta}{2} = \dfrac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$ Hence, proved.

11. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}|=3$, $|\vec{b}|=4$, $|\vec{c}|=5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Solution : Given $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$.

Also $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{0}$

$\vec{b}\cdot(\vec{c}+\vec{a}) = \vec{0}$

and $\vec{c}\cdot(\vec{a}+\vec{b}) = \vec{0}$

Since they are perpendicular

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0, \vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{a}=0$ and $\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0$

Adding all the above we get,

$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a} = 0$ ...(1)

Consider $|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$= 9+16+25+2(0) = 50$ [From (1)]

$\therefore |\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = \sqrt{25\times 2} = 5\sqrt{2}$

12. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}+6\hat{j}+3\hat{k}$. **[Hy - 2018]**

Solution : Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$

$\vec{a}\cdot\vec{b} = (\hat{i}+3\hat{j}+7\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})$

$= 1(2)+3(6)+7(3)$

$= 2+18+21 = 41$

$|\vec{b}| = \sqrt{2^2+6^2+3^2} = \sqrt{4+36+9}$

$= \sqrt{49} = 7$

Now, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{41}{7}$

13. Find λ, when the projection of $\vec{a} = \lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units.

Solution : Given $\vec{a} = \lambda\hat{i}+\hat{j}+4\hat{k}$ and

$\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$

$|\vec{b}| = \sqrt{2^2+6^2+3^2}$

$= \sqrt{4+36+9} = \sqrt{49} = 7$

$\vec{a}\cdot\vec{b} = (\lambda\hat{i}+\hat{j}+4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})$

$= 2\lambda+6+12 = 2\lambda+18$

Also projection of $\vec{a}$ on $\vec{b}$ = 4 units

We know that, projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$4 = \dfrac{2\lambda+18}{7}$

$28 = 2\lambda+18$

$28-18 = 2\lambda$

$10 = 2\lambda$

$\lambda = \frac{10}{2} = 5$

$\therefore \lambda = 5$

14. Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $|\vec{a}|=2, |\vec{b}|=3, |\vec{c}|=4$, and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Find $4\vec{a}\cdot\vec{b}+3\vec{b}\cdot\vec{c}+3\vec{c}\cdot\vec{a}$.

Solution : Given $|\vec{a}|=2, |\vec{b}|=3, |\vec{c}|=4$ and

$\vec{a}+\vec{b}+\vec{c} = \vec{0} \Rightarrow \vec{a}+\vec{b} = -\vec{c}$

$$\therefore |\vec{a}+\vec{b}|^2 = |-\vec{c}|^2$$

$$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b}) = |\vec{c}|^2$$

$$\Rightarrow 4+9+2(\vec{a}\cdot\vec{b}) = 16$$

$$\Rightarrow 13+2(\vec{a}\cdot\vec{b}) = 16$$

$$\Rightarrow 2(\vec{a}\cdot\vec{b}) = 16-13=3$$

$$\Rightarrow \vec{a}\cdot\vec{b} = \frac{3}{2}$$

$$\Rightarrow 4(\vec{a}\cdot\vec{b}) = 4\times\frac{3}{2}=6 \quad \ldots (1)$$

Also, $\vec{b}+\vec{c} = -\vec{a}$

$$|\vec{b}+\vec{c}|^2 = |-\vec{a}|^2$$

$$|\vec{b}|^2+|\vec{c}|^2+2(\vec{b}\cdot\vec{c}) = |\vec{a}|^2$$

$$9+16+2(\vec{b}\cdot\vec{c}) = 4$$

$$25+2(\vec{b}\cdot\vec{c}) = 4$$

$$2(\vec{b}\cdot\vec{c}) = 4-25=-21$$

$$(\vec{b}\cdot\vec{c}) = \frac{-21}{2}$$

$$3(\vec{b}\cdot\vec{c}) = 3\left(\frac{-21}{2}\right) = \frac{-63}{2} \quad \ldots (2)$$

Also, $\vec{c}+\vec{a} = -\vec{b}$

$$|\vec{c}+\vec{a}| = |-\vec{b}|$$

$$|\vec{c}+\vec{a}|^2 = |-\vec{b}|^2$$

$$|\vec{c}|^2+|\vec{a}|^2+2(\vec{c}\cdot\vec{a}) = |\vec{b}|^2$$

$$16+4+2(\vec{c}\cdot\vec{a}) = 9$$

$$20+2(\vec{c}\cdot\vec{a}) = 9$$

$$\Rightarrow 2(\vec{c}\cdot\vec{a}) = 9-20=-11$$

$$(\vec{c}\cdot\vec{a}) = \frac{-11}{2}$$

$$\therefore 3(\vec{c}\cdot\vec{a}) = 3\left(\frac{-11}{2}\right) = \frac{-33}{2} \quad \ldots (3)$$

Adding (1), (2) and (3) we get,

$$4\vec{a}\cdot\vec{b}+3\vec{b}\cdot\vec{c}+3\vec{c}\cdot\vec{a} = 6-\frac{63}{2}-\frac{33}{2}$$

$$= \frac{12-63-33}{2} = \frac{12-96}{2} = \frac{-84}{2} = -42$$

$$\therefore 4\vec{a}\cdot\vec{b}+3\vec{b}\cdot\vec{c}+3\vec{c}\cdot\vec{a} = -42$$

EXERCISE 8.4

1. Find the magnitude of $\vec{a}\times\vec{b}$ if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+5\hat{j}-2\hat{k}$.

Solution : Given $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$,

$$\vec{b} = 3\hat{i}+5\hat{j}-2\hat{k}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

Expanding along R_1 we get,

$$= \hat{i}(-2-15)-\hat{j}(-4-9)+\hat{k}(10-3)$$

$$\vec{a}\times\vec{b} = -17\hat{i}+13\hat{j}+7\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-17)^2+13^2+7^2}$$

$$= \sqrt{289+169+49} = \sqrt{507}$$

2. Show that $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$.

Solution : LHS $= \vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

(By associative property)

$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$$

$$\left[\because \begin{array}{l} \vec{b}\times\vec{a} = -\vec{a}\times\vec{b} \\ \vec{c}\times\vec{a} = -\vec{a}\times\vec{c} \\ \vec{c}\times\vec{b} = -\vec{b}\times\vec{c} \end{array}\right]$$

$$= \vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}=\vec{0} = \text{RHS}$$

Hence proved.

3. Find the vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane which contains $\hat{i}+2\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}+4\hat{k}$

Solution : Let $\vec{a} = \hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+3\hat{j}+4\hat{k}$

A unit vector which is perpendicular to the vector $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 1\\ 1 & 3 & 4\end{vmatrix} = \hat{i}(8-3) - \hat{j}(4-1) + \hat{k}(3-2) = 5\hat{i} - 3\hat{j} + \hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{5^2+(-3)^2+1^2} = \sqrt{25+9+1} = \sqrt{35}$$

$\therefore$ A unit vector which is perpendicular to the vector $\vec{a}$ and $\vec{b}$ is $\dfrac{5\hat{i}-3\hat{j}+\hat{k}}{\sqrt{35}}$

Hence, a vector of magnitude $10\sqrt{3}$, which is perpendicular to the vectors $\vec{a}$ and $\vec{b}$ is $\pm\dfrac{10\sqrt{3}}{\sqrt{35}}\left(5\hat{i}-3\hat{j}+\hat{k}\right)$

4. Find the unit vectors perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$. **[March - 2019]**

Solution : Given $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$

$$\therefore \vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k}$$

$$\vec{a}-\vec{b} = -\hat{j}-2\hat{k}$$

A unit vector which is perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix} = \hat{i}(-6+4)-\hat{j}(-4+0)+\hat{k}(-2+0) = -2\hat{i}+4\hat{j}-2\hat{k}$$

Its magnitude is $\sqrt{(-2)^2+4^2+(-2)^2} = \sqrt{4+16+4} = \sqrt{24} = \sqrt{4\times 6} = 2\sqrt{6}$

$\therefore$ The unit vector which is perpendicular to $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is $\pm\dfrac{(-2\hat{i}+4\hat{j}-2\hat{k})}{2\sqrt{6}} = \pm\dfrac{(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}$

5. Find the area of the parallelogram whose two adjacent sides are determined by the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

Solution : Let the adjacent sides of the parallelogram are $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b} = 3\hat{i}-2\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 3 & -2 & 1\end{vmatrix} = \hat{i}(2+6)-\hat{j}(1-9)+\hat{k}(-2-6) = 8\hat{i}+8\hat{j}-8\hat{k} = 8(\hat{i}+\hat{j}-\hat{k})$$

$$|\vec{a}\times\vec{b}| = 8\sqrt{1^2+1^2+(-1)^2} = 8\sqrt{3}$$

$\therefore$ Area of the parallelogram $= 8\sqrt{3}$ sq. units.

6. Find the area of the triangle whose vertices are A(3, – 1, 2), B(1, – 1, – 3) and C(4, – 3, 1).

Solution : Given that the vertices of the ΔABC as A(3, – 1, 2), B(1, –1, –3) and C(4, –3, 1)

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = (\hat{i}-\hat{j}-3\hat{k})-(3\hat{i}-\hat{j}+2\hat{k}) = -2\hat{i}-5\hat{k}$$

$$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = (4\hat{i}-3\hat{j}+\hat{k})-(3\hat{i}-\hat{j}+2\hat{k}) = \hat{i}-2\hat{j}-\hat{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix} = \hat{i}(0-10)-\hat{j}(2+5)+\hat{k}(4) = -10\hat{i}-7\hat{j}+4\hat{k}$$

$$|\overrightarrow{AB}\times\overrightarrow{AC}| = \sqrt{(-10)^2+(-7)^2+4^2} = \sqrt{100+49+16} = \sqrt{165}$$

Hence the required area of ΔABC $= \dfrac{1}{2}\sqrt{165}$ sq. units.

7. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the vertices A, B, C of a triangle ABC, show that the area of the triangle ABC is $\dfrac{1}{2}\left|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right|$. Also deduce the condition for collinearity of the points A, B, and C.

Solution : Given that the position vector of the vertices of the ΔABC is $\vec{a}$, $\vec{b}$ and $\vec{c}$.

$$\therefore \overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b} \text{ and } \overrightarrow{OC} = \vec{c}$$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = \vec{b}-\vec{a}$$

$$\vec{AC} = \vec{OC} - \vec{OA} = \vec{c} - \vec{a}$$

$$\therefore \vec{AB} \times \vec{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$$= \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}$$

$$= \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{0}$$

$$[\because \vec{a} \times \vec{a} = \vec{0},\ \vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}),\ \vec{a} \times \vec{c} = -(\vec{c} \times \vec{a})]$$

$$= \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

$$|\vec{AB} \times \vec{AC}| = |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

$$\therefore \text{Area of } \Delta ABC = \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$$

Condition for the points A, B, C to be collinear is area of $\Delta ABC = 0$

$$\Rightarrow \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 0$$

$\Rightarrow |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 0$ which is the required condition.

8. For any vector $\vec{a}$ prove that $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$.

Solution : Let the components of $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a} \times \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i}$$

$$= a_2(\hat{j} \times \hat{i}) + a_3(\hat{k} \times \hat{i})$$

$$= a_2(-\hat{k}) + a_3(\hat{j}) = a_3\hat{j} - a_2\hat{k}$$

$$|\vec{a} \times \hat{i}| = \sqrt{a_3^2 + (-a_2)^2} = \sqrt{a_3^2 + a_2^2}$$

$$\therefore |\vec{a} \times \hat{i}|^2 = a_3^2 + a_2^2 \quad \dots (1)$$

$$\vec{a} \times \hat{j} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{j}$$

$$= a_1(\hat{i} \times \hat{j}) + a_3(\hat{k} \times \hat{j})$$

$$= a_1\hat{k} - a_3\hat{i}$$

$$|\vec{a} \times \hat{j}| = \sqrt{a_1^2 + (-a_3)^2} = \sqrt{a_1^2 + a_3^2}$$

$$\therefore |\vec{a} \times \hat{j}|^2 = a_1^2 + a_3^2 \quad \dots (2)$$

$$\vec{a} \times \hat{k} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{k}$$

$$= a_1(\hat{i} \times \hat{k}) + a_2(\hat{j} \times \hat{k})$$

$$= -a_1\hat{j} + a_2\hat{i}$$

$$|\vec{a} \times \hat{k}| = \sqrt{(-a_1)^2 + a_2^2} = \sqrt{a_1^2 + a_2^2}$$

$$\therefore |\vec{a} \times \hat{k}|^2 = a_1^2 + a_2^2 \quad \dots (3)$$

Adding (1), (2) and (3) we get, $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2$

$$= a_3^2 + a_2^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$$

$$= 2(a_1^2 + a_2^2 + a_3^2)$$

$$= 2\left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = 2|\vec{a}|^2$$

Hence proved.

9. Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $= \frac{\pi}{3}$. Prove that $\vec{a} = \pm\frac{2}{\sqrt{3}}(\vec{b} \times \vec{c})$.

Solution : Given $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors.

$$\Rightarrow |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$$

$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0$, and angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$

$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0 \Rightarrow \vec{a}$ is $\perp^r$ to both $\vec{b}$ and $\vec{c}$.

$\vec{a}$ is $\perp^r$ to $\vec{b} \times \vec{c} \Rightarrow \vec{a} = \lambda(\vec{b} \times \vec{c})$ for some scalar λ.

$$\therefore |\vec{a}|^2 = \lambda^2 |\vec{b} \times \vec{c}|^2$$

$$\Rightarrow 1 = \lambda^2 [|\vec{b}|^2 |\vec{c}|^2 - (\vec{b} \cdot \vec{c})^2]$$

$$[\because |\vec{a}| = 1 \text{ and } |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2]$$

$$\Rightarrow 1 = \lambda^2 [(1)(1) - |\vec{b}|^2 |\vec{c}|^2 \cos^2 \frac{\pi}{3}]$$

$$[\because \text{angle between } \vec{b} \text{ and } \vec{c} \text{ is } \frac{\pi}{3}]$$

$$\Rightarrow 1 = \lambda^2 [1 - \cos^2 \frac{\pi}{3}] \quad [\because |\vec{b}| = |\vec{c}| = 1]$$

$$\Rightarrow 1 = \lambda^2 [1 - \frac{1}{4}] \Rightarrow 1 = \lambda^2 \left(\frac{3}{4}\right)$$

$$\Rightarrow \lambda^2 = \frac{4}{3} \Rightarrow \lambda = \pm\frac{2}{\sqrt{3}}$$

Substituting $\lambda = \pm\frac{2}{\sqrt{3}}$ in (1) we get,

$$\vec{a} = \pm\frac{2}{\sqrt{3}}(\vec{b} \times \vec{c})$$

10. Find the angle between the vectors $2\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ using vector product.

Solution : Let $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}(1+2)-\hat{j}(2+1)+\hat{k}(4-1)$$

$$= 3\hat{i}-3\hat{j}+3\hat{k} = 3(\hat{i}-\hat{j}+\hat{k})$$

$$|\vec{a}\times\vec{b}| = 3\sqrt{1^2+1^2+(-1)^2} = 3\sqrt{3}$$

$$|\vec{a}| = \sqrt{2^2+1^2+(-1)^2} = \sqrt{6}$$

$$|\vec{b}| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$= \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{3\sqrt{3}}{\sqrt{6}\sqrt{6}} = \frac{\not{3}\sqrt{3}}{\not{6}} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

EXERCISE 8.5

CHOOSE THE CORRECT OR THE MOST SUITABLE ANSWER FROM THE GIVEN FOUR ALTERNATIVES.

1. The value of $\vec{AB}+\vec{BC}+\vec{DA}+\vec{CD}$ is

(1) $\vec{AD}$ (2) $\vec{CA}$ (3) $\vec{0}$ (4) $-\vec{AD}$

Hint : $\vec{AB}+\vec{BC}+\vec{DA}+\vec{CD} = \vec{AB}+\vec{BC}+\vec{CD}+\vec{DA} = \vec{AA} = \vec{0}$ **[Ans : (3) $\vec{0}$]**

2. If $\vec{a}+2\vec{b}$ and $3\vec{a}+m\vec{b}$ are parallel, then the value of *m* is

(1) 3 (2) $\frac{1}{3}$ (3) 6 (4) $\frac{1}{6}$

Hint :
$$\vec{a}+2\vec{b} = 3(\vec{a}+2\vec{b})$$
$$= 3\vec{a}+6\vec{b} = 3\vec{a}+m\vec{b}$$
$$m = 6$$
[Ans: (3) 6]

3. The unit vector parallel to the resultant of the vectors $\hat{i}+\hat{j}-\hat{k}$ and $\vec{i}-2\vec{j}+\vec{k}$ is **[March - 2019]**

(1) $\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{5}}$ (2) $\frac{2\hat{i}+\hat{j}}{\sqrt{5}}$

(3) $\frac{2\hat{i}-\hat{j}+\hat{k}}{\sqrt{5}}$ (4) $\frac{2\hat{i}-\hat{j}}{\sqrt{5}}$

Hint : Resultant vector of $\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ is $2\hat{i}-\hat{j}$

Its magnitude is $\sqrt{2^2+(-1)^2} = \sqrt{4+1} = \sqrt{5}$

∴ Required unit vector = $\frac{2\hat{i}-\hat{j}}{\sqrt{5}}$ **[Ans: (4) $\frac{2\hat{i}-\hat{j}}{\sqrt{5}}$]**

4. A vector $\vec{OP}$ makes 60° and 45° with the positive direction of the *x* and *y* axes respectively. Then the angle between $\vec{OP}$ and the *z*-axis is

(1) 45° (2) 60° (3) 90° (4) 30°

Hint : Given α = 60°, β = 45°

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$$
$$\Rightarrow \cos^2 60+\cos^2 45+\cos^2\gamma = 1$$
$$\Rightarrow \left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\gamma = 1$$
$$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\gamma = 1 \Rightarrow \frac{3}{4}+\cos^2\gamma = 1$$
$$\Rightarrow \cos^2\gamma = 1-\frac{3}{4}$$
$$= \frac{1}{4} = \left(\frac{1}{2}\right)^2 = (\cos 60)^2$$
$$\Rightarrow \cos\gamma = \cos 60$$
$$\therefore \gamma = 60°$$
[Ans: (2) 60°]

5. If $\vec{BA} = 3\hat{i}+2\hat{j}+\hat{k}$ and the position vector of B is $\hat{i}+3\hat{j}-\hat{k}$ then the position vector A is

(1) $4\hat{i}+2\hat{j}+\hat{k}$ (2) $4\hat{i}+5\hat{j}$

(3) $4\hat{i}$ (4) $-4\hat{i}$

Hint :
$$\vec{BA} = 3\hat{i}+2\hat{j}+\hat{k}$$
$$\vec{OA}-\vec{OB} = 3\hat{i}+2\hat{j}+\hat{k}$$
$$\vec{OA} = 3\hat{i}+2\hat{j}+\not{\hat{k}}+\hat{i}+3\hat{j}-\not{\hat{k}} = 4\hat{i}+5\hat{j}$$
[Ans: (2) $4\hat{i}+5\hat{j}$]

6. A vector makes equal angle with the positive direction of the coordinate axes. Then each angle is equal to

(1) $\cos^{-1}\left(\frac{1}{3}\right)$ (2) $\cos^{-1}\left(\frac{2}{3}\right)$

(3) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (4) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Hint : Given $\alpha = \beta = \gamma$

$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

$\Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos^2\alpha = \frac{1}{3}$

$\Rightarrow \cos\alpha = \frac{1}{\sqrt{3}}$

$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

[Ans: (3) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$]

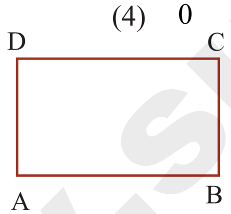
7. The vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are

(1) parallel to each other
(2) unit vectors
(3) mutually perpendicular vectors
(4) coplanar vectors. **[Ans: (4) coplanar vectors]**

8. If ABCD is a parallelogram, then $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}$ is equal to

(1) $2(\overrightarrow{AB}+\overrightarrow{AD})$ (2) $4\overrightarrow{AC}$

(3) $4\overrightarrow{BD}$ (4) $\vec{0}$



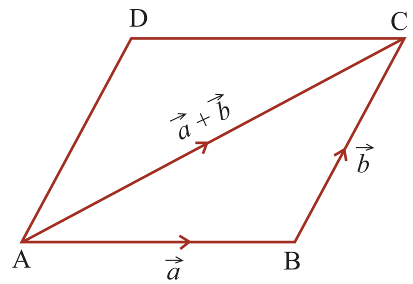
Hint : $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD} = \overrightarrow{AB}+\overrightarrow{AD}-\overrightarrow{AD}-\overrightarrow{AB} = \vec{0}$

[Ans: (4) $\vec{0}$]

9. One of the diagonals of parallelogram ABCD with $\vec{a}$ and $\vec{b}$ as adjacent sides is $\vec{a}+\vec{b}$. The other diagonal $\overrightarrow{BD}$ is

(1) $\vec{a}-\vec{b}$ (2) $\vec{b}-\vec{a}$

(3) $\vec{a}+\vec{b}$ (4) $\frac{\vec{a}+\vec{b}}{2}$



Hint : In Δ BCD, $\overrightarrow{BD} = \overrightarrow{BC}+\overrightarrow{CD} = \vec{b}-\vec{a}$

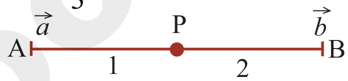
[Ans: (2) $\vec{b}-\vec{a}$]

10. If $\vec{a}, \vec{b}$ are the position vectors A and B, then which one of the following points whose position vector lies on AB, is **[March - 2019]**

(1) $\vec{a}+\vec{b}$ (2) $\frac{2\vec{a}-\vec{b}}{2}$

(3) $\frac{2\vec{a}+\vec{b}}{3}$ (4) $\frac{\vec{a}-\vec{b}}{3}$

Hint :



$\overrightarrow{OP} = \frac{1(\vec{b})+2(\vec{a})}{1+2} \Rightarrow \overrightarrow{OP} = \frac{2\vec{a}+\vec{b}}{3}$

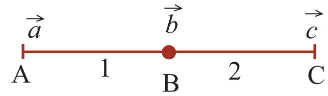
[Ans: (3) $\frac{2\vec{a}+\vec{b}}{3}$]

11. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of three collinear points, then which of the following is true?

(1) $\vec{a} = \vec{b}+\vec{c}$ (2) $2\vec{a} = \vec{b}+\vec{c}$

(3) $\vec{b} = \vec{c}+\vec{a}$ (4) $4\vec{a}+\vec{b}+\vec{c} = \vec{0}$

Hint : Since the points are collinear.



$\overrightarrow{AB} = \overrightarrow{CA} \Rightarrow \overrightarrow{OB}-\overrightarrow{OA} = \overrightarrow{OA}-\overrightarrow{OC}$

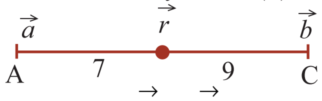
$\Rightarrow \vec{b}-\vec{a} = \vec{a}-\vec{c} \Rightarrow \vec{b}+\vec{c} = 2\vec{a}$

[Ans: (2) $2\vec{a} = \vec{b}+\vec{c}$]

12. If $\vec{r} = \dfrac{9\vec{a}+7\vec{b}}{16}$ then the point P whose position vector $\vec{r}$ divides the line joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio.

(1) 7 : 9 internally (2) 9 : 7 internally
(3) 9 : 7 externally (4) 7 : 9 externally

Hint :



Given $\vec{r} = \dfrac{9\vec{a}+7\vec{b}}{9+7}$ **[Ans: (1) 7 : 9 internally]**

13. If $\lambda\hat{i}+2\lambda\hat{j}+2\lambda\hat{k}$ is a unit vector, then the value of λ is

(1) $\frac{1}{3}$ (2) $\frac{1}{4}$ (3) $\frac{1}{9}$ (4) $\frac{1}{2}$

Hint :

$$|\lambda\hat{i}+2\lambda\hat{j}+2\lambda\hat{k}| = 1$$
$$\sqrt{\lambda^2+(2\lambda)^2+(2\lambda)^2} = 1$$
$$\Rightarrow \sqrt{\lambda^2+4\lambda^2+4\lambda^2} = 1 \Rightarrow \sqrt{9\lambda^2} = 1$$
$$\Rightarrow 3\lambda = 1 \Rightarrow \lambda = \frac{1}{3}$$

[Ans: (1) $\frac{1}{3}$]

14. Two vertices of a triangle have position vectors $3\hat{i}+4\hat{j}-4\hat{k}$ and $2\hat{i}+3\hat{j}+4\hat{k}$. If the position vector of the centroid is $\hat{i}+2\hat{j}+3\hat{k}$, then the position vector of the third vertex is

(1) $-2\hat{i}-\hat{j}+9\hat{k}$ (2) $-2\hat{i}-\hat{j}-6\hat{k}$
(3) $2\hat{i}-\hat{j}+6\hat{k}$ (4) $-2\hat{i}+\hat{j}+6\hat{k}$

Hint :

$$\overrightarrow{OA} = 3\hat{i}+4\hat{j}-4\hat{k}$$
$$\overrightarrow{OB} = 2\hat{i}+3\hat{j}+4\hat{k}$$
$$\overrightarrow{OG} = \hat{i}+2\hat{j}+3\hat{k}$$
$$\overrightarrow{OG} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$$
$$\Rightarrow 3\overrightarrow{OG} = \overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$$
$$\Rightarrow 3(\hat{i}+2\hat{j}+3\hat{k}) = (3\hat{i}+4\hat{j}-\cancel{4\hat{k}})+(2\hat{i}+3\hat{j}+\cancel{4\hat{k}})+\overrightarrow{OC}$$
$$3\hat{i}+6\hat{j}+9\hat{k} = (5\hat{i}+7\hat{j})+\overrightarrow{OC}$$
$$(3\hat{i}+6\hat{j}+9\hat{k})-(5\hat{i}+7\hat{j}) = \overrightarrow{OC}; -2\hat{i}-\hat{j}+9\hat{k} = \overrightarrow{OC}$$

[Ans: (1) $-2\hat{i}-\hat{j}+9\hat{k}$]

15. If $|\vec{a}+\vec{b}| = 60$, $|\vec{a}-\vec{b}| = 40$ and $|\vec{b}| = 46$, then $|\vec{a}|$ is

(1) 42 (2) 12 (3) 22 (4) 32

Hint : We know $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = 2[|\vec{a}|^2+|\vec{b}|^2]$

$$60^2+40^2 = 2(|\vec{a}|^2+46^2)$$
$$3600+1600 = 2(|\vec{a}|^2+2116)$$
$$\frac{5200}{2} = |\vec{a}|^2+2116$$
$$2600-2116 = |\vec{a}|^2$$
$$484 = |\vec{a}|^2$$
$$|\vec{a}| = \sqrt{484} = 22$$

[Ans: (3) 22]

16. If $\vec{a}$ and $\vec{b}$ having same magnitude and angle between them is 60° and their scalar product is $\frac{1}{2}$ then $|\vec{a}|$ is

(1) 2 (2) 3 (3) 7 (4) 1

Hint :

$$|\vec{a}| = |\vec{b}|, \theta = 60^\circ, \vec{a}\cdot\vec{b} = \frac{1}{2}$$
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$\frac{1}{2} = |\vec{a}||\vec{a}|\cos 60 \Rightarrow \cancel{\frac{1}{2}} = |\vec{a}|^2\cdot\cancel{\frac{1}{2}}$$
$$\Rightarrow |\vec{a}|^2 = 1 \Rightarrow |\vec{a}| = 1$$

[Ans: (4) 1]

17. The value of $\theta \in \left(0, \frac{\pi}{2}\right)$ for which the vectors $\vec{a} = (\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ and $\vec{b} = \hat{i}-\sqrt{3}\hat{j}+2\hat{k}$ are perpendicular, is equal to

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{2}$

Hint :

$$\vec{a}\perp\vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0$$
$$[\sin\theta\,\hat{i}+(\cos\theta)\hat{j}]\cdot[\hat{i}-\sqrt{3}\hat{j}+2\hat{k}] = 0$$
$$\sin\theta\,(1)-\sqrt{3}\cos\theta+2(0) = 0$$
$$\Rightarrow \sin\theta = \sqrt{3}\cos\theta$$
$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \sqrt{3} \Rightarrow \tan\theta = \sqrt{3}$$
$$\Rightarrow \theta = \frac{\pi}{3}$$

[Ans: (1) $\frac{\pi}{3}$]

18. If $|\vec{a}| = 13$, $|\vec{b}| = 5$ and $\vec{a}.\vec{b} = 60$ then $|\vec{a} \times \vec{b}|$ is **[Hy- 2018]**

(1) 15 (2) 35 (3) 45 (4) 25

Hint :

$$|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2 = [|\vec{a}|^2 |\vec{b}|^2]$$
$$|\vec{a} \times \vec{b}|^2 + 60^2 = [13^2 . 5^2]$$
$$|\vec{a} \times \vec{b}|^2 + 3600 = 169(25)$$
$$|\vec{a} \times \vec{b}|^2 = 4225 - 3600 = 625$$
$$|\vec{a} \times \vec{b}| = \sqrt{625} = 25$$

[Ans: (4) 25]

19. Vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle $\theta = 120°$. If $|\vec{a}| = 1$, $|\vec{b}| = 2$, then $[(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})]^2$ is equal to

(1) 225 (2) 275 (3) 325 (4) 300

Hint :

$$[(\vec{a} + 3\vec{b}) \times (3\vec{a} - \vec{b})]^2 = [\vec{a} \times 3\vec{a} - \vec{a} \times \vec{b} + 9\vec{b} \times \vec{a} - 3\vec{b} \times \vec{b}]^2$$
$$= [0 - \vec{a} \times \vec{b} - 9\vec{a} \times \vec{b} - 0]^2 \quad [\because \vec{a} \times \vec{a} = \vec{b} \times \vec{b} = \vec{0}]$$
$$= [-10\vec{a} \times \vec{b}]^2 = 100|\vec{a} \times \vec{b}|^2 = 100.\ [|\vec{a}|^2 |\vec{b}|^2 \sin^2\theta]$$
$$= 100[(1)^2 (2)^2 \sin^2 120] = 100 \times 4 \times [\sin(180 - 60)]^2$$
$$= 400\ [\sin 60]^2 = 400 \times \left(\frac{\sqrt{3}}{2}\right)^2 = \overset{100}{\cancel{400}} \times \frac{3}{\cancel{4}} = 300$$

[Ans: (4) 300]

20. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 2 and inclined at an angle 60°, then the angle between $\vec{a}$ and $\vec{a} + \vec{b}$ is

(1) 30° (2) 60° (3) 45° (4) 90°

Hint : $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = 2^2 + 2^2 + 2|\vec{a}|.|\vec{b}|$

$$\cos\theta = 4 + 4 + 2(2)(2)(\cos 60)$$
$$= 8 + \overset{4}{\cancel{8}}\left(\frac{1}{\cancel{2}}\right) = 8 + 4 = 12$$
$$\therefore |\vec{a} + \vec{b}|^2 = \sqrt{12} = 2\sqrt{3}$$

Let α be the angle between $\vec{a}$ and $\vec{a} + \vec{b}$

$$\therefore \cos\alpha = \frac{\vec{a} \cdot (\vec{a} + \vec{b})}{|\vec{a}||\vec{a} + \vec{b}|} = \frac{\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b}}{|\vec{a}||\vec{a} + \vec{b}|}$$
$$= \frac{|\vec{a}|^2 + \vec{a} \cdot \vec{b}}{|\vec{a}||\vec{a} + \vec{b}|} = \frac{2^2 + |\vec{a}||\vec{b}|\cos\theta}{2(2\sqrt{3})}$$
$$= \frac{4 + 2(\cancel{2})\left(\frac{1}{\cancel{2}}\right)}{4\sqrt{3}} = \frac{4+2}{4\sqrt{3}} = \frac{\overset{3}{\cancel{6}}}{\underset{2}{\cancel{4}}\sqrt{3}}$$
$$= \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \alpha = 30^o$$

[Ans: (1) 30°]

21. If the projection of $5\hat{i} - \hat{j} - 3\hat{k}$ on the vector $\hat{i} + 3\hat{j} + \lambda\hat{k}$ is same as the projection of $\hat{i} + 3\hat{j} + \lambda\hat{k}$ on $5\hat{i} - \hat{j} - 3\hat{k}$ then λ is equal to

(1) ± 4 (2) ± 3 (3) ± 5 (4) ±1

Hint : Let $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 3\hat{j} + \lambda\hat{k}$, $\vec{c} = \hat{i} + 3\hat{j} + \lambda\hat{k}$,
$\vec{d} = 5\hat{i} - \hat{j} - 3\hat{k}$

Given projection of $\vec{a}$ on $\vec{b}$ = projection of $\vec{c}$ on $\vec{d}$

$$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{\vec{c} \cdot \vec{d}}{|\vec{d}|}$$
$$\Rightarrow \frac{5(1) - 1(3) - 3(\lambda)}{\sqrt{1^2 + 3^2 + \lambda^2}} = \frac{5 - 3 - 3\lambda}{\sqrt{5^2 + (-1)^2 + (-3)^2}}$$
$$\Rightarrow \frac{2 - 3\lambda}{\sqrt{10 + \lambda^2}} = \frac{2 - 3\lambda}{\sqrt{25 + 1 + 9}}$$
$$\sqrt{10 + \lambda^2} = \sqrt{35}$$

[Equating the denominator]

Squaring, $10 + \lambda^2 = 35 \Rightarrow \lambda^2 = 25$

$\Rightarrow \lambda = \pm 5$ **[Ans: (3) ± 5]**

22. If (1, 2, 4) and (2, – 3λ, – 3) are the initial and terminal points of the vector $\hat{i} + 5\hat{j} - 7\hat{k}$, then the value of λ is equal to

(1) $\frac{7}{3}$ (2) $-\frac{7}{3}$ (3) $-\frac{5}{3}$ (4) $\frac{5}{3}$

Hint : Given $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 4\hat{k}$ and $\overrightarrow{OB} = 2\hat{i} - 3\lambda\hat{j} - 3\hat{k}$

and $\overrightarrow{AB} = \hat{i} + 5\hat{j} - 7\hat{k}$

But $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

$$\hat{i} + 5\hat{j} - 7\hat{k} = (2\hat{i} - 3\lambda\hat{j} - 3\hat{k}) - (\hat{i} + 2\hat{j} + 4\hat{k})$$
$$\hat{i} + 5\hat{j} - 7\hat{k} = \hat{i} + (-3\lambda - 2)\hat{j} - 7\hat{k}$$

Equating the like components both sides, we get

$$5 = -3\lambda - 2 \Rightarrow 7 = -3\lambda$$

$\Rightarrow \quad \lambda = \frac{-7}{3}$ **[Ans: (2) $\lambda = \frac{-7}{3}$]**

23. If the points whose position vectors $10\hat{i}+3\hat{j}, 12\hat{i}-5\hat{j}$ and $a\hat{i}+11\hat{j}$ are collinear then a is equal to

(1) 6 (2) 3 (3) 5 (4) 8

Hint : $\vec{OA} = 10\hat{i}+3\hat{j}$; $\vec{OB} = 12\hat{i}-5\hat{j}$ and $\vec{OC} = a\hat{i}+11\hat{j}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (12\hat{i}-5\hat{j})-(10\hat{i}+3\hat{j}) = 2\hat{i}-8\hat{j}$

$\vec{BC} = (a-12)\hat{i}+16\hat{j}$

$\vec{CA} = (10-a)\hat{i}-8\hat{j}$

$\vec{AB} = \vec{CA} \Rightarrow 2\hat{i}-8\hat{j} = (10-a)\hat{i}-8\hat{j}$

$\Rightarrow \quad 2 = 10-a$ [Equating $\hat{i}$ components]

$\Rightarrow \quad a = 10-2 = 8$ **[Ans: (4) 8]**

24. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+x\hat{j}+\hat{k}$, $\vec{c} = \hat{i}-\hat{j}+4\hat{k}$ and $\vec{a}\cdot(\vec{b}\times\vec{c}) = 70$, then x is equal to

(1) 5 (2) 7 (3) 26 (4) 10

Hint : $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+x\hat{j}+\hat{k}$, $\vec{c} = \hat{i}-\hat{j}+4\hat{k}$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & x & 1 \\ 1 & -1 & 4 \end{vmatrix}$$

$= \hat{i}(4x+1)-\hat{j}(8-1)+\hat{k}(-2-x)$

$= \hat{i}(4x+1)+\hat{j}(-7)+\hat{k}(-2-x)$

Given $\vec{a}\cdot(\vec{b}\times\vec{c}) = 70$

$\Rightarrow 1(4x+1)+1(-7)+1(-2-x) = 70$

$\Rightarrow 4x+1-7-2-x = 70$

$\Rightarrow 3x-8 = 70$

$\Rightarrow 3x = 78$

$\Rightarrow x = \frac{78}{3} = 26$ **[Ans: (3) 26]**

25. If $\vec{a} = \hat{i}+2\hat{j}+2\hat{k}$, $|\vec{b}| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then the area of the triangle formed by these two vectors as two sides, is

(1) $\frac{7}{4}$ (2) $\frac{15}{4}$ (3) $\frac{3}{4}$ (4) $\frac{17}{4}$

Hint : Given $\vec{a} = \hat{i}+2\hat{j}+2\hat{k}$, $|\vec{b}| = 5$ angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$.

$|\vec{a}| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

Area of the triangle formed by $\vec{a}$ and $\vec{b}$

$= \frac{1}{2}|\vec{a}\times\vec{b}| = \frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$

$= \frac{1}{2}\left[3(5)\sin\frac{\pi}{6}\right] = \frac{1}{2}\left[15\times\frac{1}{2}\right] = \frac{15}{4}$

[Ans: (2) $\frac{15}{4}$]

ADDITIONAL PROBLEMS

SECTION - A (1 MARK)

1. If $m\left(\vec{2}+\vec{j}+\vec{k}\right)$ is a unit vector then the value of m is **[Hy - 2018]**

(1) $\pm\frac{1}{\sqrt{3}}$ (2) $\pm\frac{1}{\sqrt{5}}$ (3) $\pm\frac{1}{\sqrt{6}}$ (4) $\pm\frac{1}{2}$

Hint : $m\left(\vec{2}+\vec{j}+\vec{k}\right)$ is a unit vector

$\left|m\left(\vec{2}+\vec{j}+\vec{k}\right)\right| = 1$

$|m|\left|\vec{2}+\vec{j}+\vec{k}\right| = 1$

$|m|\sqrt{2^2+1^2+(-1)^2} = 1$

$|m|\sqrt{6} = 1$

$|m| = \frac{1}{\sqrt{6}}$

$|m| = \pm\frac{1}{\sqrt{6}}$ **[Ans:(3) $\pm\frac{1}{\sqrt{6}}$]**

2. If $\vec{a}, \vec{b}$ are the position vectors of A and B, then which one of the following points whose position vector lies on AB? **[March - 2019]**

(1) $\frac{2\vec{a}+\vec{b}}{3}$ (2) $\frac{\vec{a}-\vec{b}}{3}$

(3) $\vec{a}+\vec{b}$ (4) $\frac{2\vec{a}-\vec{b}}{2}$

Hint : $\vec{OA} = 2\hat{i}+5\hat{j}$

$\vec{OB} = 5\hat{i}+7\hat{j}+4\hat{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = 5\hat{i}+2\hat{j}+4\hat{k}$

[Ans: (3) $-5\hat{i}+2\hat{j}+4\hat{k}$]

3. The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4) respectively is

(1) $-\hat{i}+12\hat{j}+4\hat{k}$ (2) $5\hat{i}+2\hat{j}-4\hat{k}$

(3) $-5\hat{i}+2\hat{j}+4\hat{k}$ (4) $\hat{i}+\hat{j}+\hat{k}$

Hint :

$\vec{OA} = 2\hat{i}+5\hat{j}$

$\vec{OB} = 5\hat{i}+7\hat{j}+4\hat{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = 5\hat{i}+2\hat{j}+4\hat{k}$

[Ans: (3) $-5\hat{i}+2\hat{j}+4\hat{k}$ **]**

4. The value of λ when the vectors $\vec{a} = 2\vec{i}+\lambda\vec{j}+\vec{k}$ and $\vec{b} = \vec{i}+2\vec{j}+3\vec{k}$ are orthogonal is

(1) 0 (2) 1 (3) $\frac{3}{2}$ (4) $-\frac{5}{2}$

Hint :

$\vec{a}\cdot\vec{b} = 2(1)+\lambda(2)+(1)3 = 0$

$\Rightarrow 2+2\lambda+3 = 0$

$\lambda = \frac{-5}{2}$ **[Ans:** (4) $-\frac{5}{2}$ **]**

5. The value of *m* for which the vectors $3\hat{i}-6\hat{j}+\hat{k}$ and $2\hat{i}-4\hat{j}+\lambda\hat{k}$ are parallel is

(1) $\frac{2}{3}$ (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{2}{5}$

Hint :

$3\hat{i}-6\hat{j}+\hat{k} = \frac{3}{2}\left(2\hat{i}-4\hat{j}+\lambda\hat{k}\right)$

$= 3\hat{i}-6\hat{j}+\frac{3\lambda}{2}\hat{k}$

$\frac{3\lambda}{2} = 1 \Rightarrow \lambda = \frac{2}{3}$ **[Ans:** (1) $\frac{2}{3}$ **]**

6. Match List - I with List II

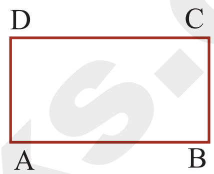
	List I		List II
i.	$\hat{i}.\hat{i}$	(a)	0
ii.	$\hat{i}.\hat{j}$	(b)	$\hat{k}$
iii.	$\hat{i}\times\hat{i}$	(c)	1
iv.	$\hat{i}\times\hat{j}$	(d)	0

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	b	c	d	a
(2)	c	a	d	b
(3)	d	b	a	c
(4)	d	c	b	a

[Ans : (2) i – c ii – a iii – d iv – b**]**

7. Assertion (A) : If ABCD is a prallelogram, $\vec{AB}+\vec{AD}+\vec{CB}+\vec{CD}$ then is equal to zero.



Reason (R) : $\vec{AB}$ and $\vec{CD}$ are equal in magnitude and opposite in direction. Also $\vec{AD}$ and $\vec{CB}$ are equal in magnitude and opposite in direction

(1) Both A and R are true and R is the correct explanation of A
(2) Both A and R are true and R is not a correct explantion of A
(3) A is true but R is false
(4) A is false but R is true

[Ans: (1) Both A and R are true and R is the correct explanation of A**]**

8. Find the odd one out of the following

(1) $\hat{i}+2\hat{j}+3\hat{k}$ (2) $2\hat{i}+4\hat{j}+6\hat{k}$

(3) $7\hat{i}+14\hat{j}+21\hat{k}$ (4) $\hat{i}+3\hat{j}+2\hat{k}$

Hint : (1), (2), (3) are parallel vectors

[Ans: (4) $\hat{i}+3\hat{j}+2\hat{k}$ **]**

9. Assertion (A) : $\vec{a},\vec{b},\vec{c}$ are the position vector of three collinear points then $2\vec{a} = \vec{b}+\vec{c}$

Reason (R) : Collinear points, have same direction

(1) Both A and R are true and R is the correct explanation of A
(2) Both A and R are true and R is not a correct explantion of A
(3) A is true but R is false
(4) A is false but R is true

[Ans: (1) Both A and R are true and R is the correct explanation of A**]**

10. Find the odd one out of the following

(1) matrix multiplication
(2) vector cross product
(3) Subtraction
(4) Matrix Addition

Hint : Only (4) is commutative

[Ans: (4)Matrix Addition]

SECTION - B (2 MARKS)

1. Define diagonal and scalar matrices.[March - 2019]

Solution : Diagonal; In a square matrix $A = [a_{ij}]_{n \times r}$ Of order n, the elements $a^{11}, a^{22}, a^{33} \ldots.. a^{nn}$ are called the principal diagonal or simply the diagonal

Scalar matrix:

A diagonal matrix whose entries along the principal diagonal are equal is called a scalar matrix.

2. Find a unit vector along the direction of the vector $5\hat{i} - 3\hat{j} + 4\hat{k}$ [March - 2019]

Solution : $\vec{a} = 5\hat{i} - 3\hat{j} + 4\hat{k}$

$$\therefore \hat{a} = \pm \frac{\vec{a}}{|\vec{a}|} = \pm \frac{(5\hat{i} - 3\hat{j} + 4\hat{k})}{5\sqrt{2}}$$

3. If $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 4\hat{j} + \hat{k}$ then find $|\vec{a} - 2\vec{b}|$.

Solution :

Given $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} - 4\hat{j} + \hat{k}$

$\therefore \vec{a} - 2\vec{b} = (3\hat{i} - 2\hat{j} + \hat{k}) - 2(2\hat{i} - 4\hat{j} + \hat{k})$

$= -\hat{i} + 6\hat{j} - \hat{k}$

$|\vec{a} - 2\vec{b}| = \sqrt{(-1)^2 + 6^2 + (-1)^2}$

$= \sqrt{1+36+1} = \sqrt{38}$

4. Write two different vectors having same magnitude.

Solution : Let $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ be two vectors.

Then, $|\vec{a}| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$

and $|\vec{b}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$

Hence the required vectors are $2\hat{i} - \hat{j} + 3\hat{k}$ and $\hat{i} + 2\hat{j} - 3\hat{k}$.

5. Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (–5, 7).

Solution : Let A(2, 1) be initial point and B(–5, 7) be terminal point of given vector.

Then, $\overrightarrow{AB} = (-5-2)\hat{i} + (7-1)\hat{j} = -7\hat{i} + 6\hat{j}$

$\therefore$ The scalar components of $\overrightarrow{AB}$ are –7 and 6.

The vector components of $\overrightarrow{AB}$ are $-7\hat{i}$ and $6\hat{j}$.

6. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ are $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.

Solution : Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$

Then $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 4^2}$

$= \sqrt{4+9+16} = \sqrt{29}$

and $|\vec{b}| = \sqrt{(-4)^2 + 6^2 + (-8)^2}$

$= \sqrt{16+36+64} = \sqrt{116}$

$= \sqrt{4 \times 29} = 2\sqrt{29}$

$\therefore |\vec{b}| = 2|\vec{a}|$

Thus, $\vec{a}$ and $\vec{b}$ are collinear.

7. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$ then find $\vec{a} \times \vec{b}$. Verify that $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

Solution : Given $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - 5\hat{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 3 & -5 \end{vmatrix}$$

$= \hat{i}(-10-9) - \hat{j}(-5-6) + \hat{k}(3-4)$

$= -19\hat{i} + 11\hat{j} - \hat{k}$

Now, $\vec{a} \cdot (\vec{a} \times \vec{b}) = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-19\hat{i} + 11\hat{j} - \hat{k})$

$= 1(-19) + 2(11) + 3(-1)$

$= -19 + 22 - 3 = -\cancel{22} + \cancel{22} = 0$

This shows that $\vec{a}$ and $(\vec{a} \times \vec{b})$ are perpendicular to each other.

SECTION - C (3 MARKS)

1. Find the unit vector in the direction of the vector $\vec{a} - 2\vec{b} + 3\vec{c}$ if $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{k}$.

Solution : Given now, $\vec{a} = \hat{i} + \hat{j}$; $\vec{b} = \hat{j} + \hat{k}$; $\vec{c} = \hat{i} + \hat{k}$

$$\therefore \vec{a} - 2\vec{b} + 3\vec{c} = (\hat{i} + \hat{j}) - 2(\hat{j} + \hat{k}) + 3(\hat{i} + \hat{k})$$
$$= 4\hat{i} - \hat{j} + \hat{k}$$
$$\therefore |\vec{a} - 2\vec{b} + 3\vec{c}| = \sqrt{4^2 + (-1)^2 + 1^2} = \sqrt{16+1+1}$$
$$= \sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}$$

Thus, the unit vector in the direction of $\vec{a} - 2\vec{b} + 3\vec{c}$ is

$$\frac{\vec{a} - 2\vec{b} + 3\vec{c}}{|\vec{a} - 2\vec{b} + 3\vec{c}|} = \frac{1}{3\sqrt{2}}(4\hat{i} - \hat{j} + \hat{k})$$

2. Find the direction cosines of the vector joining the points A(1, 2, –3) and B(–1, –2, 1) directed from A to B.

Solution : Given points are A (1, 2, –3) and B (–1, – 2, 1).

$$\text{Then } \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (-\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k})$$
$$= -2\hat{i} - 4\hat{j} + 4\hat{k}$$
$$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2}$$
$$= \sqrt{4+16+16} = \sqrt{36} = 6$$
$$\text{Now, } l = \frac{x}{|\overrightarrow{AB}|} = \frac{-2}{6} = \frac{-1}{3}$$
$$m = \frac{y}{|\overrightarrow{AB}|} = \frac{-4}{6} = \frac{-2}{3}$$
$$n = \frac{z}{|\overrightarrow{AB}|} = \frac{4}{6} = \frac{2}{3}$$

Thus, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

3. Find $|\vec{x}|$ if for a unit vector $\vec{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$

Solution : Given $|\vec{a}| = 1$ and $(\vec{x} - \vec{a})(\vec{x} + \vec{a}) = 12$

$$\Rightarrow \vec{x} \cdot \vec{x} - \vec{a} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{a} = 12$$
$$\Rightarrow |\vec{x}|^2 - \cancel{\vec{a} \cdot \vec{x}} + \cancel{\vec{a} \cdot \vec{x}} - |\vec{a}|^2 = 12$$
$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12 \quad [\because |\vec{a}| = 1]$$
$$\Rightarrow |\vec{x}|^2 = 13$$
$$\Rightarrow |\vec{x}| = \sqrt{13}$$

4. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-coplanar vectors. Let A, B and C be the points whose position vectors with respect to the origin O are $\vec{a} + 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} + 5\vec{c}$ and $7\vec{a} - \vec{c}$ respectively. Then prove that A, B and C are collinear.

Solution : Given $\overrightarrow{OA} = \vec{a} + 2\vec{b} + 3\vec{c}$

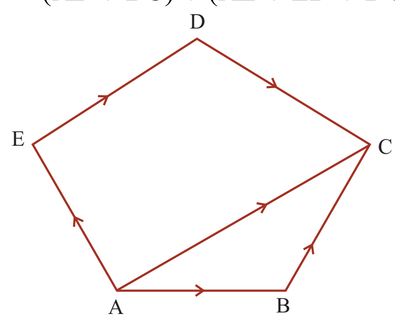
$$\overrightarrow{OB} = -2\vec{a} + 3\vec{b} + 5\vec{c} \text{ and } \overrightarrow{OC} = 7\vec{a} - \vec{c}$$
$$\text{Then } \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (-2\vec{a} + 3\vec{b} + 5\vec{c}) - (\vec{a} + 2\vec{b} + 3\vec{c})$$
$$= -3\vec{a} + \vec{b} + 2\vec{c}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (7\vec{a} - \vec{c}) - (\vec{a} + 2\vec{b} + 3\vec{c})$$
$$= 6\vec{a} - 2\vec{b} - 4\vec{c}$$
$$= -2(-3\vec{a} + \vec{b} + 2\vec{c}) = -2\overrightarrow{AB}$$

$\therefore \overrightarrow{AC} \parallel \overrightarrow{AB}$ and A is a common points. Hence, the points A, B and C are collinear.

5. If ABCDE is a pentagaon then prove that $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC} = 3\overrightarrow{AC}$

Solution : $\overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$$= (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$$



$$= \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$$

(Using triangle law of addition)

$$= 3\overrightarrow{AC} \text{ Hence proved.}$$

SECTION - D (5 MARKS)

1. Let $\vec{a} = 2\vec{j} + \vec{j} - 2\vec{k}$; $\vec{b} = 2\vec{i} + \vec{j}$. If $\vec{c}$ is a vector such that $\vec{a}\cdot\vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ is 30º. Find the value of $\left|(\vec{a}\times\vec{b})\times\vec{c}\right|$ **[Hy - 2018]**

Solution :

$$\vec{a}\cdot\vec{b} = 3 \Rightarrow |\vec{c}| = 3$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\vec{i} - 2\vec{j} + \vec{k}$$

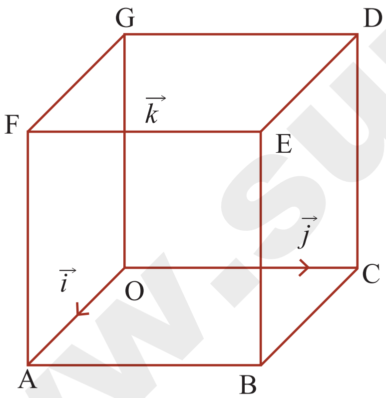
$$|\vec{c}\times\vec{i}| = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\left|(\vec{a}\times\vec{i})\times\vec{c}\right| = \left|(\vec{a}\times\vec{i})\right|\times|\vec{c}|\ \sin 30^{\circ}$$

$$= 3\times 3\times\frac{1}{2} = \frac{9}{2}$$

2. Prove that the smaller angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

Solution : Let OABCDEFG be a unit cube.



Keeping O as origin,

Let $\vec{OA} = \hat{i}$, $\vec{OC} = \hat{j}$ and $\vec{OG} = \hat{k}$

Consider the diagonals OE and BG.

$$\vec{OE} = \vec{OB} + \vec{BE} = \vec{OA} + \vec{AB} + \vec{BE}$$

$$= \vec{OA} + \vec{OC} + \vec{OG} = \hat{i} + \hat{j} + \hat{k}$$

$$[\because \vec{AB} = \vec{OC}, \vec{BE} = \vec{OG}]$$

and $\vec{GB} = \vec{GO} + \vec{OB} = -\hat{k} + \vec{OA} + \vec{AB} = \hat{i} + \hat{j} - \hat{k}$

Let θ be the smaller angle between the diagonals OE and GB, then

$$\cos\theta = \frac{\vec{OE}\cdot\vec{GB}}{|\vec{OE}||\vec{GB}|} = \frac{1(1)+1(1)+1(-1)}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{1^2+1^2+(-1)^2}}$$

$$= \frac{2-1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$$

$$\text{Thus } \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

3. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = \sqrt{24}$ and sum of any two vectors is orthogonal to the third vector, then find $|\vec{a} + \vec{b} + \vec{c}|$.

Solution : Given $(\vec{a} + \vec{b})\cdot\vec{c} = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} = 0$

$(\vec{b} + \vec{c})\cdot\vec{a} = 0$

$\Rightarrow \quad \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a} = 0$

$(\vec{c} + \vec{a})\cdot\vec{b} = 0$

$\Rightarrow \quad \vec{c}\cdot\vec{b} + \vec{a}\cdot\vec{b} = 0$

Adding, $2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = 0 \quad \ldots (1)$

$|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$

$= 9 + 16 + 24 + 2(0)$

$= 49$

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = 7$

4. If $|\vec{a}| = |\vec{b}| = |\vec{a}+\vec{b}| = 1$ then prove that $|\vec{a}-\vec{b}| = \sqrt{3}$.

Solution :

$$\text{Given } |\vec{a}+\vec{b}| = 1$$
$$|\vec{a}+\vec{b}|^2 = 1$$
$$|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}.\vec{b}) = 1$$
$$1+1+2|\vec{a}||\vec{b}|\cos\theta = 1$$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

$$2+2(1)(1)\cos\theta = 1$$
$$2\cos\theta = 1-2 = -1$$
$$\cos\theta = -\frac{1}{2}$$
$$\text{Consider } |\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}.\vec{b})$$
$$= 1+1-2|\vec{a}||\vec{b}|\cos\theta$$
$$= 2-\cancel{2}(1)(1)\left(-\frac{1}{\cancel{2}}\right)$$
$$= 2+1 = 3$$
$$\therefore |\vec{a}-\vec{b}| = \sqrt{3}$$

❋❋❋❋❋

POINTS TO REMEMBER

In this chapter we have acquired the knowledge of the following :

- A scalar is a quantity that is determined by its magnitude.
- A vector is a quantity that is determined by both its magnitude and its direction
- If we have a liberty to choose the origins of the vector at any point then it is said to be a **free vector**, whereas if it is restricted to a certain specified point then the vector is said to be a **localized vector.**
- Two or more vectors are said to be **coplanar** if they lie on the same plane or parallel to the same plane.
- Two vectors are said to be equal if they have equal length and the same direction.
- A vector of magnitude 0 is called the **zero vector.**
- A vector of magnitude 1 is called a **unit vector**.
- Let a $\vec{a}$ be a vector and m be a scalar. Then the vector $m\vec{a}$ is called the scalar multiple of a vector $\vec{a}$ by the scalar m.
- Two vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if $\vec{a} = \lambda\vec{b}$, where λ is a scalar.
- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the sides of a triangle taken in order then $\vec{a} + \vec{b} + \vec{c} = 0$
- Vector addition is associative.
- For any vector $\vec{a}$, $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.
- For any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$
- Vector addition is commutative.
- "If two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order". This is known as the **triangle law of addition.**
- In a parallelogram OABC if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represents two adjacent sides, then the diagonal $\overrightarrow{OC}$ represents their sum. **This is parallelogram law of addition.**
- If α,β,γ are the direction angles then $\cos\alpha,\cos\beta,\cos\gamma$ are the direction cosines.
- The direction ratios of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are x,y,z.
- If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors in the space, then any vector in the space can be written as $l\vec{a} + m\vec{b} + n\vec{c}$ in a unique way.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point and let α, β, γ be the direction angles of $\vec{r}$. Then

(i) the sum of the squares of the direction cosines of $\vec{r}$ is 1.

(ii) $\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2$.

(iii) the direction cosines of $\vec{r}$ are $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$

(iv) if l, m, n are the direction cosines of a vector if and only if $l^2+m^2+n^2 = 1$.

(v) any unit vector can be written as $\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$.

- The **scalar product** of the vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

- **Vector product** of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$. Here $\vec{a}$, $\vec{b}$, $\hat{n}$ form a right handed system.

Govt. Model Question Paper - 1 (21.08.2018)
with Answer Key

Govt. Model Question Paper - 2 (23.02.2019)
with Answer Key

Sura's Model Question Paper - (2 Nos.)
with Answer Key

11th STD.

Mathematics

Time : 2½ hours **Written Exam Marks : 90 Marks**

On 21.08.2018, Model Question Paper is released by the Govt.
We have given it along with Answer Key.

Kind Attention to the Students

- From this year onwards, blue print system has been abolished.
- Please note that questions will be framed from IN-TEXT portions ALSO.
- Approximately 20% of the questions will be asked from IN-TEXT portions.
- These questions will be based on Reasoning and Understanding of the lessons.
- Further, Creative and Higher Order Thinking Skills questions will also be asked. It requires the students to clearly understand the lessons. So the students have to think and answer such questions.
- It is instructed that henceforth if any questions are asked from 'out of syllabus', grace marks will not be given.
- Term Test, Revision Test and Model Exam will be conducted based on the above pattern only.
- Concentrating only on the book-back questions and/or previous year questions, henceforth, may not ensure to score 100% marks.
- Also note that the answers must be written either in blue ink or in black ink. Avoid using both the colour inks to answer the questions.
- For MCQs, the answers should be written in full. Simply writing (a) or (b) etc. will not get full marks. You have to write (a) or (b) etc., along with the answer given in the options.

On 21.08.2018, Model Question Paper is released by the Govt. We have given it along with Answer Key.

11th STD.

Govt. Model Question Paper - 1 (2018-19)

(with Answer Key)

Time : 2.30 Hours | **Mathematics** | **Maximum Marks : 90**

Section - I

Note : (i) Answer all are compulsory the questions. **[20 × 1 = 20]**

(ii) Choose the correct or most suitable answer from the given four alternatives. Write the option code and the corresponding answer.

1. If A = $\{(x, y) : y = \sin x,\ x \in R\}$ and B = $\{(x, y) : y = \cos x,\ x \in R\}$ then $A \cap B$ contains

(1) no element
(2) infinitely many elements
(3) only one element
(4) cannot be determined.

2. The number of relations on a set containing 3 elements is

(1) 9 (2) 81 (3) 512 (4) 1024

3. The function $f : [0, 2\pi] \to [-1, 1]$ defined by $f(x) = \sin x$ is

(1) one-to-one (2) onto
(3) bijection (4) cannot be defined

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1 - |x|$. Then the range of f is

(1) $\mathbb{R}$ (2) $(1,\infty)$ (3) $(-1,\infty)$ (4) $(-\infty, 1]$

5. If quadratic with real coefficients has no real roots, then its discriminant is ________equation.

(1) 0 (2) < 0 (3) >0 (4) 1

6. If $|x + 2| \le 9$, then x belongs to

(1) $(-\infty,-7)$ (2) $[-11, -7]$
(3) $(-\infty,-7)\cup[11,\infty)$ (4) $(-11,7)$

7. If a and b are the roots of the equation $x^2 - kx + 16 = 0$ and satisfy $a^2 + b^2 = 32$, then the value of k is

(1) 10 (2) -8 (3) $-8, 8$ (4) 6

8. If $\sqrt{x+14} < 2$, then x belongs to

(1) $[-14, -10)$ (2) $(-14, -10)$
(3) $(-\infty, -10)$ (4) $[-14, -10]$

9. $\cos 1^\circ + \cos 2^\circ + \cos 3^\circ + ... + \cos 179^\circ$ is

(1) 0 (2) 1 (3) -1 (4) 89

10. Which one of the following is not true for any θ?

(1) $\sin\theta = -\frac{3}{4}$ (2) $\cos\theta = -1$
(3) $\tan\theta = 25$ (4) $\sec\theta = \frac{1}{4}$

11. A wheel is spinning at 2 radians/second. How many seconds will it take to make 10 complete rotations?

(1) 10π seconds (2) 20π seconds
(3) 5π seconds (4) 15π seconds

12. $\dfrac{\sin 10^\circ - \cos 10^\circ}{\cos 10^\circ + \sin 10^\circ}$ is

(1) $\tan 35^\circ$ (2) $\sqrt{3}$ (3) $\tan 75^\circ$ (4) 1

13. The product of r consecutive positive integers is divisible by

(1) $r!$ (2) $(r-1)!$
(3) $(r+1)!$ (4) r^r

14. The number of sides of a polygon having 44 diagonals is

(1) 4 (2) 4! (3) 11 (4) 22

15. If ${}^nC_4$, ${}^nC_5$, ${}^nC_6$ are in AP then value of n is

(1) 14 (2) 11 (3) 9 (4) 5

16. The sum of the digits in the unit's place of all the 4- digit numbers formed by 3, 4, 5 and 6, without repetition, is ——.

(1) 432 (2) 108 (3) 36 (4) 72

17. If a is the arithmetic mean and g is the geometric mean of two numbers, then

(1) $a \le g$ (2) $a \ge g$ (3) $a = g$ (4) $a > g$

18. The coefficient of x^8y^{12} in the expansion of $(2x + 3y)^{20}$ is

(1) 0 (2) $2^8 3^{12}$
(3) $2^8 3^{12} + 2^{12} 3^8$ (4) ${}^{20}C_8\, 2^8 3^{12}$

19. The value of $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + ...$ is

(1) $\dfrac{e^2+1}{2e}$ (2) $\dfrac{(e+1)^2}{2e}$ (3) $\dfrac{(e-1)^2}{2e}$ (4) $\dfrac{e^2+1}{2e}$

20. The remainder when 52^{40} is divided by 17 is

(1) 1 (2) 3 (3) 5 (4) 6

Section - II

(i) Answer any **SEVEN** questions.

(ii) Question number **30** is compulsory. **7 × 2 = 14**

21. If A = {1, 2, 3, 4} and B = {3, 4, 5, 6}, find $n((A\cup B) \times (A\cap B) \times (A\Delta B))$.

22. In the set $\mathbb{Z}$ of integers, define mRn if $m - n$ is a multiple of 12. Prove that $\mathbb{R}$ is an equivalence relation.

23. If A × A has 9 elements, $S=\{(a, b) \in A \times A : a > b\}$; (2, –1) and (2, 1) are two elements, then find the remaining elements of S.

24. Prove $\log a + \log a^2 + \log a^3 + \ldots + \log a^n = \frac{n(n+1)}{2}\log a$

25. Solve : $(x - 2)(x + 3)^2 < 0$.

26. If A + B = 45°, show that (1 + tan A) (1 + tan B) = 2.

27. Prove that $\frac{\sin 4x+\sin 2x}{\cos 4x+\cos 2x} = \tan 3x$

28. Out of 6 consonants and 4 vowels, how many strings of 3 consonants and 2 vowels can be formed?

29. Prove that $\sum_{k=1}^{n}\frac{1}{k(k+1)}=1-\frac{1}{n+1}$.

30. Prove that $\log_4{}^2 - \log_8{}^2 + \log_6{}^2 - \ldots$ is $1 - \log_e{}^2$.

Section - III

(i) Answer any **SEVEN** questions.

(ii) Question number **40** is compulsory. **7 × 3 = 21**

31. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, prove that f is a bijection and find its inverse.

32. Using the given curve $y = x^3$. Draw the graph, $y = (x + 1)^3$ with the same scale.

33. If one root of $k(x - 1)^2 = 5x - 7$ is double the other root, show that $k = 2$ or -25.

34. Resolve into partial fractions: $\frac{10x+30}{(x^2-9)(x+7)}$.

35. Suppose that a boat travels 10 km from the port towards east and then turns 60° to its left. If the boat travels further 8 km, how far from the port is the boat?

36. If $A + B + C = \frac{\pi}{2}$, prove sin 2A + sin 2B + sin 2C = 4 cos A cos B cos C.

37. How many different selections of 5 books can be made from 12 different books if,

(i) Two particular books are always selected?

(ii) Two particular books are never selected?

38. How many numbers are there between 100 and 500 with the digits 0, 1, 2, 3, 4, 5 if repetition of digits is not allowed.

39. Find the co-efficient of x^{15} in the expansion $\left(x^2+\frac{1}{x^3}\right)^{10}$.

40. In ΔABC, if $\tan\frac{A}{2}=\frac{5}{6}$ and $\tan\frac{C}{2}=\frac{2}{5}$, then show that a, b, c, are in A.P.

Section - IV

Answer *all* questions. **7 × 5 = 35**

41. (a) Show that the range of the function $\frac{1}{2\cos x-1}$ is $\left(-\infty,-\frac{1}{3}\right]\cup[1,\infty)$

(OR)

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x - |x|$ and $g(x) = 2x + |x|$. Find $f \circ g$.

42. (a) Prove that the solution of $\frac{x+1}{x+3}<3$ is $(-\infty, -4)\cup(-3, \infty)$.

(OR)

(b) Determine the region in the plane determined by the inequalities $2x + 3y \le 35$, $y \ge 2$, $x \ge 5$.

43. (a) If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$, then prove that $xy + yz + zx = 0$.

(OR)

(b) Solve $\sqrt{3}\tan^2\theta + (\sqrt{3}-1)\tan\theta - 1 = 0$

44. (a) If the letters of the word APPLE are permuted in all possible ways and the strings then formed are arranged in the dictionary order show that the rank of the word APPLE is 12.

(OR)

(b) A van has 8 seats. It has two seats in the front with two rows of three seats behind. The van belongs to a family, consisting of seven members, F, M, S_1, S_2, S_3, D_1, D_2. How many ways can the family sit in the van if

(i) There are no restriction?

(ii) Either F or M drives the van?

(iii) D_1, D_2 sits next to a window and F is driving?

45. (a) Using Mathematical induction, show that for any natural number n,

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$

(OR)

(b) Prove that $\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

46. (a) Find the sum up to the 17th term of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$

(OR)

(b) Show that $\frac{5}{1\times3}+\frac{5}{2\times4}+\frac{5}{3\times5}+.... = \frac{15}{4}$

47. (a) Let A = {2, 3, 5} and relation R = {(2,5)} write down the minimum number of ordered pairs to be included to R to make it an equivalence relation.

(OR)

(b) If $x=\sum_{n=0}^{\infty}\cos^{2n}\theta, y=\sum_{n=0}^{\infty}\sin^{2n}\theta$ and

$z=\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta, 0<\theta<\frac{\pi}{2}$, then show that $xyz = x+y+z$.

ANSWERS

SECTION - I

1. (2) infinitely many elements
2. (3) 512
3. (2) onto
4. (4) $(-\infty, 1]$
5. (2) < 0
6. (2) $[-11, -7]$
7. (3) $-8, 8$
8. (1) $[-14, -10)$
9. (1) 0
10. (4) $\sec\theta = \frac{1}{4}$
11. (1) 10π seconds
12. (1) $\tan 35°$
13. (1) $r!$
14. (3) 11
15. (1) 14
16. (2) 108
17. (2) $a \geq g$
18. (4) $^{20}C_8\ 2^8 3^{12}$
19. (3) $\frac{(e-1)^2}{2e}$
20. (1) 1

SECTION - II

21. Solution : We have $n(A\cup B) = 6$,

$n(A\cap B) = 2$ and

$n(A\,\Delta\,B) = 4$

So, $n((A\cup B)\times n(A\cap B)\times(A\Delta B)) = n(A\cup B)\times n(A\cap B)\times n(A\Delta B) = 6\times2\times4 = 48$

22. Solution :

As $m - m = 0$ and $0 = 0\times12$, hence $m\mathbb{R}m$ proving that $\mathbb{R}$ is reflexive.

Let $m\mathbb{R}n$. Then $m - n = 12k$ for some integer k; thus $n - m = 12(-k)$ and hence $n\mathbb{R}m$. This shows that $\mathbb{R}$ is symmetric.

Let $m\mathbb{R}n$ and $n\mathbb{R}p$; then $m - n = 12k$ and $n - p = 12l$ for some integers k and l.

So $m - p = 12(k + l)$ and hence $m\mathbb{R}p$. This shows that $\mathbb{R}$ is transitive. Thus $\mathbb{R}$ is an equivalence relation.

23. Solution : $n(A\times A) = 9$

$\Rightarrow n(A) = 3;\ S = \{(a,b)\in A\times A : a > b\}$

$A = \{-1, 1, 2\}$

$A\times A = \{(-1,-1), (-1,1), (-1,2), (1,-1), (1,1), (1,2), (2,-1), (2,1), (2,2)\}$

$\therefore S = \{(1,-1), (2,-1), (2,1)\}$

$\therefore$ Remaining element of S is $(1, -1)$,

24. Solution :

$$\begin{aligned}\text{LHS} &= \log a + \log a^2 + \log a^3 + \ldots + \log a^n\\ &= \log a + 2\log a + 3\log a + \ldots + n\log a\\ &= \log a\,(1+2+3+\ldots.n)\\ &= \log a\,\frac{(n)(n+1)}{2} \qquad \left[\because \sum n = \frac{n(n+1)}{2}\right]\\ &= \frac{n(n+1)}{2}\log a = \text{RHS} \quad \text{Hence proved.}\end{aligned}$$

25. Solution : $(x-2)(x+3)^2 < 0$

Critical numbers 2, – 3

We have three intervals $(-\infty,-3), (-3,2), (2,\infty)$

Interval	Sign of $(x-2)$	Sign of $(x+3)^2$	Sign of $(x-2)(x+3)$
$(-\infty,-3)\ x=-5$	–	+	–
$(-3,2)\ x=0$	–	+	–
$(2,\infty)\ x=3$	+	+	+

The inequality is satisfied in the interval $(-\infty,-3)$ and $(-3, 2)$

$\therefore$ Solution set is $(-\infty,-3)\cup(-3,2)$

26. Solution :

Given $A + B = 45° \Rightarrow B = 45° - A$

$$\begin{aligned}\text{LHS} &= (1+\tan A)(1+\tan B)\\ &= (1+\tan A)(1+\tan(45°-A))\\ &= (1+\tan A)\left(1+\frac{\tan 45° - \tan A}{1+\tan 45°\cdot\tan A}\right)\end{aligned}$$

$$\left[\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\ \tan B}\right]$$

$$\begin{aligned}&= (1+\tan A)\left(1+\frac{1-\tan A}{1+\tan A}\right)\\ &= (1+\tan A)\left(\frac{1+\tan A+1-\tan A}{1+\tan A}\right)\\ &= (1+\tan A)\frac{2}{1+\tan A} = 2 = \text{RHS}\end{aligned}$$

Hence proved.

27. Solution : LHS $= \dfrac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x}$

$$= \frac{\cancel{2}\sin\left(\frac{4x+2x}{2}\right)\cdot\cos\left(\frac{4x-2x}{2}\right)}{\cancel{2}\cos\left(\frac{4x+2x}{2}\right)\cos\left(\frac{4x-2x}{2}\right)} = \frac{\sin 3x\cdot\cancel{\cos x}}{\cos 3x\cdot\cancel{\cos x}} = \tan 3x = \text{RHS.}$$

Hence proved.

28. Solution :

Number of ways of selecting (3 consonants out of 6) and (2 vowels out of 4) is $6C_3 \times {}^4C_2$

Each string contains 5 letters. Number of ways of arranging 5 letters among themselves is $5! = 120$. Hence required number of ways is

$6C_3 \times {}^4C_2 \times 5! = 20 \times 6 \times 120 = 14400$

29. Solution :

Let t_k denote the k^{th} term of the given series. Then

$t_k = \dfrac{1}{k(k+1)}$. By using partial fraction we get

$\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ Thus $t_1 + t_2 + ... + t_n$

$$\left(1-\frac{1}{\cancel{2}}\right)+\left(\frac{1}{\cancel{2}}-\frac{1}{\cancel{3}}\right)+\left(\frac{1}{\cancel{3}}-\frac{1}{\cancel{4}}\right)+...+\left(\frac{1}{\cancel{n}}-\frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

30. Solution :

$$\begin{aligned}
\text{LHS} &= \log_4 2 - \log_8 2 + \log_{16} 2 - \log_{32} 2 + ...\\
&= \frac{1}{\log_2 4} - \frac{1}{\log_2 8} + \frac{1}{\log_2 16} - \frac{1}{\log_2 32} + ...\\
&= \frac{1}{\log_2 2^2} - \frac{1}{\log_2 2^3} + \frac{1}{\log_2 2^4} - \frac{1}{\log_2 2^5} + ...\\
&= \frac{1}{2\log_2 2} - \frac{1}{3\log_2 2} + \frac{1}{4\log_2 2} - \frac{1}{5\log_2 2} + ...\\
&= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + ...\\
&= 1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + ...\\
&= 1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\right) = 1 - \log_e 2 = \text{RHS}
\end{aligned}$$

SECTION - III

31. Solution : Let $y = 3x - 5$.

$\Rightarrow \quad y + 5 = 3x \Rightarrow \dfrac{y+5}{3} = x.$

Let $g(y) = \dfrac{y+5}{3}$.

$$\begin{aligned}
gof(x) &= g(f(x)) = g(3x-5)\\
&= \frac{3x - \cancel{5} + \cancel{5}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x
\end{aligned}$$

$$\begin{aligned}
\text{Also } fog(y) &= f(g(y)) = f\left(\frac{y+5}{\cancel{3}}\right)\\
&= \cancel{3}\left(\frac{y+5}{\cancel{3}}\right) - 5 = y + \cancel{5} - \cancel{5} = y.
\end{aligned}$$

Thus $gof = \text{I}x$ and $fog = \text{I}y$.

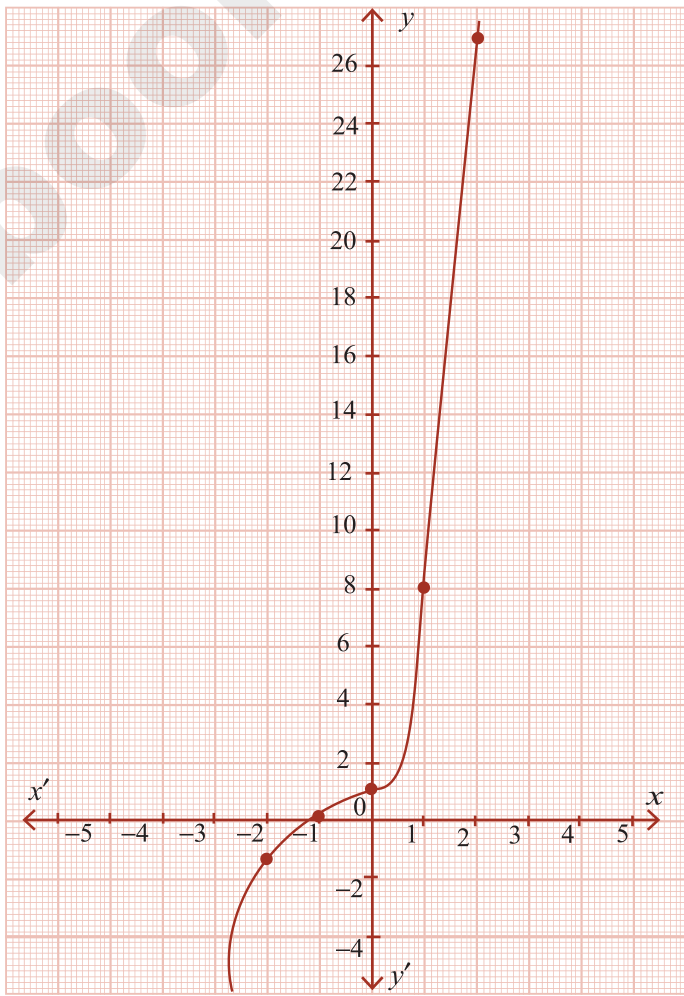
This implies that f and g are bijections and inverses to each other.

Hence f is a bijection and $f^{-1}(y) = \dfrac{y+5}{3}$.

Replacing y by x we get, $f^{-1}(x) = \dfrac{x+5}{3}$

32. Solution : $y = (x+1)^3$

x	0	1	–1	2	–2
y	1	8	0	27	–1



Let $f(x) = x^3$

$y = (x+1)^3$, causes the graph of $f(x)$ shifts to the left for one unit

33. Solution : Given equation is $k(x-1)^2 = 5x - 7$

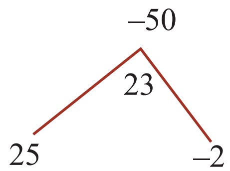
$$\Rightarrow \quad k(x^2 - 2x + 1) = 5x - 7$$
$$\Rightarrow \quad kx^2 - 2kx + k - 5x + 7 = 0$$
$$\Rightarrow \quad kx^2 + x(-2k - 5) + (k + 7) = 0$$

Let the roots be α and 2α.

$$\therefore \alpha + 2\alpha = \frac{+2k+5}{k}$$
$$\Rightarrow \quad 3\alpha = \frac{+2k+5}{k}$$
$$\Rightarrow \quad \alpha = \frac{+2k+5}{3k} \text{ and } \alpha(2\alpha) = \frac{k+7}{k}$$
$$\Rightarrow \quad 2\alpha^2 = \frac{k+7}{k}$$
$$\Rightarrow \quad \alpha^2 = \frac{k+7}{2k} \qquad ...(2)$$

Substituting (1) in (2) we get,

$$\left(\frac{2k+5}{3k}\right)^2 = \frac{k+7}{2k}$$



$$\Rightarrow \quad \frac{4k^2+25+20k}{9k^2} = \frac{k+7}{2k}$$
$$\Rightarrow \quad \frac{4k^2+25+20k}{9k} = \frac{k+7}{2}$$
$$\Rightarrow \quad 8k^2 + 50 + 40k = 9k^2 + 63k$$
$$\Rightarrow \quad k^2 + 23k - 50 + 0 = 0$$
$$\Rightarrow \quad (k-2)(k+25) = 0$$
$$\Rightarrow \quad k = 2 \text{ or } -25.$$

Hence proved.

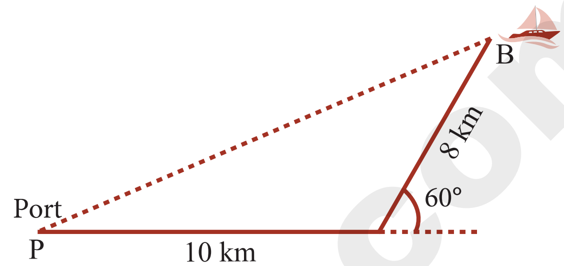
34. Solution : Let $\dfrac{10x+30}{(x^2-9)(x+7)}$

$$= \frac{10(x+3)}{(x+3)(x-3)(x+7)} = \frac{10}{(x-3)(x+7)}$$
$$= \frac{A}{x-3} + \frac{B}{x+7} = \frac{A(x+7)+B(x-3)}{(x-3)(x+7)}$$
$$\therefore 10 = A(x+7) + B(x-3)$$
$$x = 3 \text{ then } A = 1$$
$$x = -7 \text{ then } B = -1$$

Hence, $$\frac{10x+30}{(x^2-9)(x+7)} = \frac{1}{x-3} - \frac{1}{x+7}$$

35. Solution : Let BP be the required distance. By using the cosine formula, we have,

$$BP^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos 120^\circ$$
$$= 244 \text{ km} \Rightarrow BP = 2\sqrt{61} \text{ km}$$



36. Solution :

LHS = $\sin 2A + \sin 2B + \sin 2C$

$= 2\sin\left(\dfrac{2A+2B}{2}\right)\cos\left(\dfrac{2A-2B}{2}\right) + 2\sin C\cos C.$

$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$= 2\sin\left(\dfrac{\pi}{2} - C\right)\cos(A-B) + 2\sin C\cos C$

$= 2\cos C\cos(A-B) + 2\sin C\cos C$

$= 2\cos C\left[\cos(A-B) + \sin\left(\dfrac{\pi}{2} - (A+B)\right)\right]$

$= 2\cos C \cdot 2\cos A\cos B = 4\cos A\cos B\cos C$ = RHS

Hence proved.

37. Solution :

(i) Two particular books are always selected

Two particular books are always selected, the remaining 3 books can be selected from 10 books in $^{10}C_3$ ways.

$$^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120 \text{ ways}$$

(ii) Two particular books are never selected

Since two books are never to be selected, the selection of 5 books from 10 books are done in $10C_5$ ways.

$$= \frac{10!}{5!\,5!} = \frac{10\times9\times8\times7\times6\times5!}{5!\;5\times4\times3\times2\times1} = \frac{10\times9\times8\times7\times6}{5\times4\times3\times2} = 252.$$

38. Solution : Repetition of digit is not allowed

4	6	6

Hundreds place can be filled in 4 ways excluding 0 and 5.

Tens place can be filled in 5 ways since repetition of digits is not allowed.

Unit place can be filled in 4 ways.

∴ By fundamental principle of multiplication, required number of three digit numbers = 4 × 5 × 4 = 80

39. Solution : In $\left(x^2+\frac{1}{x^3}\right)^{10}$, $n = 10, x = x^2, a = \frac{1}{x^3}$,

So the general term is $T_{r+1} = {}^nCr\, x^{n-r}\, a^r$

$$\Rightarrow T_{r+1} = {}^{10}C_r (x^2)^{10-r}\left(\frac{1}{x^3}\right)^r$$
$$= {}^{10}C_r\, x^{20-2r}.$$
$$x^{-3r} = {}^{10}C_r\, x^{20-5r} \quad ...(1)$$

To find the Co-efficient of x^{15},

put $20 - 5r = 15$

$\Rightarrow 20 - 15 = 5r \Rightarrow 5 = 5r \Rightarrow r = 1$

putting $r = 1$ in (1) we get

$T_2 = {}^{10}C_1 x^{20-5} = {}^{10}C_1 x^{15}$

$\therefore$ Co-efficient of x^{15} is 10.

40. Solution :

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad ...(1)$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \quad ...(2)$$

$$\left(\tan\frac{A}{2}\right)\left(\tan\frac{C}{2}\right) = \sqrt{\frac{(s-b)\cancel{(s-c)}}{s\cancel{(s-a)}}\times\frac{(s-b)\cancel{(s-a)}}{s\cancel{(s-c)}}} = \frac{s-b}{s} \quad ... (3)$$

Also, $\left(\tan\frac{A}{2}\right)\left(\tan\frac{C}{2}\right) = \frac{\cancel{5}}{\cancel{6}_3}\times\frac{\cancel{2}}{\cancel{5}} = \frac{1}{3}$

$$\therefore \frac{s-b}{s} = \frac{1}{3}$$
$$3s - 3b = s$$
$$2s = 3b$$
$$\cancel{2}\left(\frac{a+b+c}{\cancel{2}}\right) = 3b$$
$$a+b+c = 3b$$
$$a+c = 2b$$

$\therefore a, b, c$ are in A.P

SECTION - IV

41. (a) **Solution :** Range of cosine function is

$-1 \le \cos x \le 1$.

$\Rightarrow -2 \le 2\cos x \le 2$ (Multiplied by 2)

$\Rightarrow -2\ -1 \le 2\cos x - 1 \le 2 - 1$

$\Rightarrow -3 \le 2\cos x - 1 \le 1$

$\Rightarrow \frac{-1}{3} > \frac{1}{2\cos x - 1} > \frac{1}{1} \quad \Rightarrow \quad \frac{-1}{3} > f(x) > 1$

$\therefore$ Range of $f(x)$ is $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$

(OR)

(b) **Solution :** We know $|x| = \begin{cases} -x, & x \le 0 \\ x, & x > 0 \end{cases}$

So $f(x) = \begin{cases} 2x-(-x) & \text{if } x \le 0 \\ 2x - x & \text{if } x > 0 \end{cases}$

Thus $f(x) = \begin{cases} 3x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$

Also $g(x) = \begin{cases} 2x+(-x) & \text{if } x \le 0 \\ 2x + x & \text{if } x > 0 \end{cases}$

Thus $g(x) = \begin{cases} x; & x \le 0 \\ 3x; & x > 0 \end{cases}$

Let $x \le 0$. Then $(gof)(x) = g(f(x)) = g(3x) = 3x$

The last equality is taken because $3x \le 0$ whenever $x \le 0$.

Let $x > 0$. Then $(gof)(x) = g(f(x)) = g(x) = 3x$

Thus $(gof)(x) = 3x$ for all x.

42.(a) **Solution :** Subtracting 3 from both sides we get $\frac{x+1}{x+3} - 3 < 0$.

$$\frac{x+1-3(x+3)}{x+3} < 0 \Rightarrow \frac{-2x-8}{x+3} < 0 \Rightarrow \frac{x+4}{x+3} > 0$$

Thus, $x + 4$ and $x + 3$ are both positive or both negative.

So let us find out the signs of $x + 3$ and $x + 4$ as follows

x	$x+3$	$x+4$	$\frac{x+4}{x+3}$
$x < -4$	–	–	+
$-4 < x < -3$	–	+	–
$x > -3$	+	+	+
$x = -4$	–	0	0

So the solution set is given by $(-\infty, -4) \cup (-3, \infty)$.

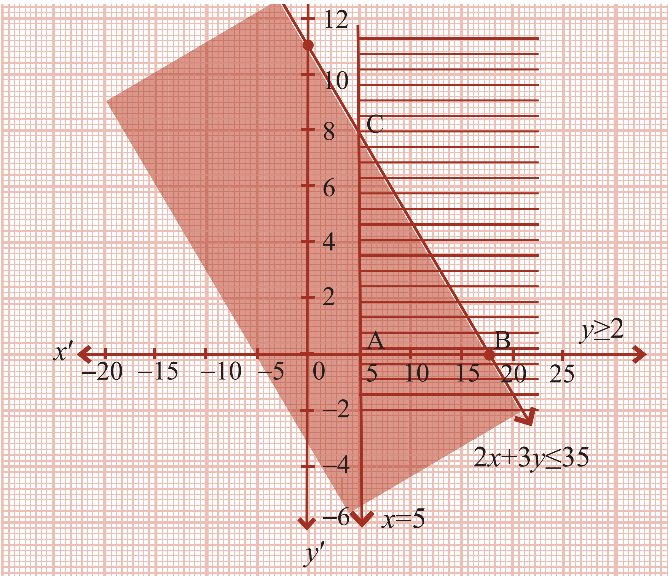
(OR)

(b) **Solution :** If $2x + 3y = 35$ then

x	0	17.5
y	11.6	0

$y = 2$ is a line parallel to X-axis at a distance 2 units.

$x = 5$ is a line parallel to Y-axis at a distance of 5 units.



ABC is the required shaded region.

43. (a) **Solution :**

Let $x \cos\theta = y\cos\left(\theta+2\dfrac{\pi}{3}\right) = z\cos\left(\theta+4\dfrac{\pi}{3}\right) = \lambda$

$\Rightarrow \dfrac{\lambda}{x} = \cos\theta, \dfrac{\lambda}{y} = \cos\left(\theta+2\dfrac{\pi}{3}\right)$ and $\dfrac{\lambda}{z} = \cos\left(\theta+4\dfrac{\pi}{3}\right)$

$\therefore \dfrac{\lambda}{x}+\dfrac{\lambda}{y}+\dfrac{\lambda}{z} = \cos\theta+\cos\left(\theta+2\dfrac{\pi}{3}\right)+\cos\left(\theta+4\dfrac{\pi}{3}\right)$

$= \cos\theta + \cos(120+\theta) + \cos(240+\theta)$

$= \cos\theta + \cos 120 \cos\theta - \sin 120 \sin\theta + \cos 240 \cos\theta - \sin 240 \sin\theta$

$= \cos\theta - \cos 60 \cos\theta - \sin 60 \sin\theta - \sin 30 \cos\theta + \cos 30 \sin\theta$

$\cos\theta - \dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta - \dfrac{1}{2}\cos\theta + \dfrac{\sqrt{3}}{2}\sin\theta$

$= \cos\theta - \dfrac{1}{2}\cos\theta - \dfrac{1}{2}\cos\theta = \cos\theta - \cos\theta = 0$

$\therefore \lambda\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) = 0 \Rightarrow \lambda\left(\dfrac{yz+xz+xy}{xyz}\right) = 0$

$\Rightarrow \quad xy + yz + zx = 0$

[$\because \cos(A+B) = \cos A\cos B - \sin A \sin B$

$\cos 120^\circ = \cos(180^\circ - 60^\circ)$

$= -\cos 60^\circ = \dfrac{-1}{2}$

$\sin 240^\circ = \sin(270^\circ - 30^\circ)$

$= \sin 60$

$= -\cos 30 = \dfrac{-\sqrt{3}}{2}$]

(OR)

(b) $\sqrt{3}\tan^2\theta + (\sqrt{3}-1)\tan\theta - 1 = 0$

$\sqrt{3}\tan^2\theta + \sqrt{3}\tan\theta - \tan\theta - 1 = 0$

$(\sqrt{3}\tan\theta - 1)(\tan\theta + 1) = 0$

Thus, either $\sqrt{3}\tan\theta - 1 = 0$ (or)

$\tan\theta + 1 = 0$

If $\sqrt{3}\tan\theta - 1 = 0$, then

$\tan\theta = \dfrac{1}{\sqrt{3}} = \tan\dfrac{\pi}{6}$

$\Rightarrow \theta = n\pi + \dfrac{\pi}{6}, n\in\mathbb{Z}$ ... (i)

If $\tan\theta + 1 = 0$ then

$\tan\theta = -1 = \tan\left(\dfrac{-\pi}{4}\right)$

$\Rightarrow \theta = n\pi - \dfrac{\pi}{4}, n\in\mathbb{Z}$ ... (ii)

From (i) and (ii) we have the general solution.

44. (a) **Solution :** In the word APPLE, there are 5 letters.

The lexicographic order of the word is A, E, L, P, P

The letter P occurs 2 times.

Number of words starting with AE $= \dfrac{3!}{2} = 3$

Number of words starting with AL $= \dfrac{3!}{2} = 3$

Number of words starting with APE $= 2! = 2$

Number of words starting with APL $= 2! = 2$

APPEL $= 1$

APPLE $= 1$

Rank of APPLE $= 3 + 3 + 2 + 2 + 1 + 1 = 12$

Hence proved.

(OR)

(b) **Solution :**

(i) As there 8 seats to be occupied out of which one seat is for the one who drives. Since there are no restrictions any one can drive the van. Hence the number of ways of occupying the driver seat is $^7P_1$ = 7 ways . The number of ways of occupying the remaining 7 seats by the remaining 6 people is $^7P_6$ = 5040: Hence the total number of ways the family can be seated in the car is 7 × 5040 = 35280:

(ii) As the driver seat can be occupied by only F or M, there are only two ways it can be occupied. Hence the total number of ways the family can be seated in the car is 2 × 5040 = 10080:

(iii) As there are only 5 window seats available for D_1 & D_2 to occupy the number of ways of seated near the windows by the two family members is $^5P_2$ = 20. As the driver seat is occupied by F, the remaining 4 people can be seated in the available 5 seats in $^5P_4$ = 120: Hence the total number of ways the family can be seated in the car is 20 × 1 × 120 = 2400:

45. (a) **Solution :**

Let $P(n) = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)} = \frac{n}{n+1}$.

Substituting the value of n = 1, in the statement we get,

$$P(1) = \frac{1}{1.2} = \frac{1}{2}.$$

Hence P(1) is true. Let us assume that the statement is true for $n = k$. Then

$$P(k) = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)} = \frac{k}{k+1}$$

We need to show that P(k + 1) is true. Consider,

$$P(k+1) = \underbrace{\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

$$= P(k)+\frac{1}{(k+1)(k+2)} = \frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{k+1}\left(\frac{k}{1}+\frac{1}{k+2}\right) = \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$= \frac{1}{k+1}\left(\frac{(k+1)^2}{k+2}\right) = \frac{(k+1)}{(k+2)}$$

This implies, P(k + 1) is true. The validity of P(k + 1) follows from that of P(k). Therefore, by the principle of mathematical induction, for any natural number n,

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+...+\frac{1}{n(n+1)} = \frac{n}{n+1}$$

(OR)

(b) **Solution :** $\sqrt[3]{x^3+7} = (x^3+7)^{\frac{1}{3}} = \left[x^3\left(1+\frac{7}{x^3}\right)\right]^{\frac{1}{3}}$

$$\left(\left|\frac{7}{x^3}\right| < 1 \text{ as } x \text{ is large}\right)$$

$$= x\left(1+\frac{7}{x^3}\right)^{\frac{1}{3}} = x\left(1+\frac{1}{3}\times\frac{7}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2+...\right)$$

$$= x\left(1+\frac{7}{3}\times\frac{1}{x^3}-\frac{49}{9}\times\frac{1}{x^6}+...\right)$$

$$= x+\frac{7}{3}\times\frac{1}{x^2}-\frac{49}{9}\times\frac{1}{x^5}+... = \sqrt[3]{x^3+4} = (x^3+4)^{\frac{1}{3}}$$

$$= \left[x^3\left(1+\frac{4}{x^3}\right)\right]^{\frac{1}{3}} = x\left(1+\frac{4}{x^3}\right)^{\frac{1}{3}} \quad \left(\left|\frac{4}{x^3}\right| < 1\right)$$

$$= x\left(1+\frac{1}{3}\times\frac{4}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{4}{x^3}\right)^2+...\right)$$

$$= x+\frac{4}{3}\times\frac{1}{x^2}-\frac{16}{9}\times\frac{1}{x^6}+...$$

Since x is large, $\frac{1}{x}$ is very small and hence higher powers $\frac{1}{x}$ of are negligible. Thus $\sqrt[3]{x^3+7}$ $= x+\frac{7}{3}\times\frac{1}{x^2}$ and $\sqrt[3]{x^3+4} = x+\frac{4}{3}\times\frac{1}{x^2}$ .Therefore

$$\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4} = \left(\cancel{x}+\frac{7}{3}\times\frac{1}{x^2}\right)-\left(\cancel{x}+\frac{4}{3}\times\frac{1}{x^2}\right) = \frac{1}{x^2}$$

46. (a) **Solution :** Let T_n be the n^{th} term of the given series

$$\because T_n = \frac{1^3+2^3+...+n^3}{1+3+5+...+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{\frac{n}{2}(1+2n-1)} \left[\because S_n = \frac{n}{2}(a+l)\right] =$$

$$\frac{n^2(n+1)^2}{4} \Big/ \frac{n}{\cancel{2}}(\cancel{2}n)$$

$$= \frac{n^2(n+1)^2}{4}\times\frac{1}{n^2} = \frac{(n+1)^2}{4} = \frac{1}{4}(n^2+2n+1)$$

Let S_n denote the sum of n terms of the given series.

Then

$$S_n = \sum_{k=1}^{n} T_k = \frac{1}{4}\Sigma(k^2+2k+1) = \frac{1}{4}\left[\sum_{k=1}^{n}k^2+2\sum_{k=1}^{n}k+\sum_{k=1}^{n}1\right]$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}+\frac{2(n)(n+1)}{2}+n\right]$$

$$= \frac{1}{24}\left[n(n+1)(2n+1)+6(n)(n+1)+6n\right]$$

$$= \frac{1}{24}\left[(n^2+n)(2n+1)+6n^2+6n+6n\right]$$

$$= \frac{1}{24}\left[2n^3+n^2+2n^2+n+6n^2+12n\right]$$

$$S_n = \frac{1}{24}\left[2n^3+9n^2+13n\right] = \frac{n}{24}\left[2n^2+9n+13\right]$$

Now we have to find S_{17}

$$\therefore S_{17} = \frac{17}{24}\left[2(17)^2+9(17)+13\right]$$

$$= \frac{17}{24}[578+153+13] = \frac{17}{24}(744) = 17(31) = 527.$$

$S_{17} = 527$

(OR)

(b) **Solution :** Given series can be written as $\sum_{n=1}^{\infty} \frac{5}{n(n+2)}$

Let n^{th} term be $a^n \therefore a_n = \frac{5}{n(n+2)}$

By partial fraction,

$$\frac{5}{n(n+2)} = \frac{5}{2n} - \frac{5}{2(n+2)}$$

Sum of the n terms of the series be S_n

$$\therefore S_n = a_1 + a_2 + \dots + a_n$$

$$= \left(\frac{5}{2}-\frac{5}{6}\right)+\left(\frac{5}{4}-\frac{5}{8}\right)+\left(\frac{5}{6}-\frac{5}{10}\right)+\dots+\left(\frac{5}{2(n-1)}-\frac{5}{2(n+1)}\right)+\left(\frac{5}{2n}-\frac{5}{2(n+2)}\right)$$

$$= \frac{5}{2}+\frac{5}{4}-\frac{5}{2(n+1)}-\frac{5}{2(n+2)}$$

As n tends to infinity, $\frac{5}{2(n+1)}$ and $\frac{5}{2(n+2)}$

$\frac{5}{2} = \frac{5}{4} - \frac{5}{2(n+1)} - \frac{5}{2(n+2)}$ tends to $\frac{5}{2}+\frac{5}{4} = \frac{15}{4}$ or $S_n \to \frac{15}{4}$

That is, $\frac{5}{1\times 3}+\frac{5}{2\times 4}+\frac{5}{3\times 4}+\dots = \frac{15}{4}$

47. (a) **Solution :** It is enough to add (2, 2), (3, 3) and (5, 5) to make R reflexive.

To make R symmetric, (5, 2) needs to be included.

Given R is a transitive relation.

$\therefore$ Minimum number of ordered pairs required are (5, 2), (2,2), (3,3) and (5, 5)

(OR)

(b) **Solution :**

Given $x = \sum_{n=0}^{\infty} \cos^{2n}\theta = \cos^0\theta + \cos^2\theta + \cos^4\theta + \cdots$

$= 1 + \cos^2\theta + \cos^4\theta + \dots + = 1 + (\cos\theta)^2 + (\cos^2\theta)^2 + \dots$

$$= \frac{1}{1-\cos^2\theta} \qquad \left[\because 1+x+x^2+\dots x = \frac{1}{1-x} \text{ when } |x|<1\right]$$

$$= \frac{1}{\sin^2\theta} \qquad \dots(1)$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\theta = \sin^0\theta + \sin^2\theta + \sin^4\theta + \dots$$

$= 1 + \sin^2\theta + \sin^4\theta + \dots = 1 + (\sin\theta)^2 + (\sin^2\theta)^2 + \dots$

$$= \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta} \qquad \dots(2)$$

Similarly $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\theta$

$= \cos^0\theta \sin^0\theta + \cos^2\theta \sin^2\theta + \cos^4\theta \sin^4\theta + \dots$

$$= 1 + (\sin\theta\cos\theta)^2 + (\sin^2\theta\cos^2\theta)^2 + \dots + = \frac{1}{1-\sin^2\theta\cos^2\theta} \qquad \dots(3)$$

Now $x+y+z = \frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta}$

[using (1), (2) and (3)]

$$= \frac{\cos^2\theta(1-\sin^2\theta\cos^2\theta)+\sin^2\theta(1-\sin^2\theta\cos^2\theta)+\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{\cos^2\theta-\sin^2\theta\cos^4\theta+\sin^2\theta-\sin^4\theta\cos^2\theta+\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1-(\sin^2\theta\cos^4\theta+\sin^4\theta\cos^2\theta)+\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)}$$

$[\because \sin^2\theta+\cos^2\theta = 1]$

$$= \frac{1-\sin^2\theta\cos^2\theta(\cos^2\theta+\sin^2\theta)+\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta\,(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1-\sin^2\theta\cos^2\theta(1)+\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta\,(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{\sin^2\theta}\times\frac{1}{\cos^2\theta}\times\frac{1}{1-\sin^2\theta\cos^2\theta} \quad \therefore x+y+z = xyz$$

Hence proved.

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